Inferring Expected Runtimes for Probabilistic Integer Programs Using Expected Sizes

27th Conference on Tools and Algorithms for the Construction and Analysis of Systems

Fabian Meyer **Marcel Hark** Jürgen Giesl

Randomization in Programming

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  $\to$ Measure of efficiency: expected runtime.

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  $\rightarrow$ Measure of efficiency: expected runtime.

  $\rightarrow$ How to infer upper bounds on expected runtime fully automatically?

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  Consider $\mathfrak{r} = 2 \cdot x$.

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$$\tfrac{1}{2} \cdot \mathfrak{r}\,[x/x] + \tfrac{1}{2} \cdot \mathfrak{r}\,[x/x-1] = 2 \cdot x - 1 = \mathfrak{r} - 1$$

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  Expected runtime of loop: at most $\mathfrak{r} = 2 \cdot x$. (e.g., [Bournez & Garnier '05]).

```
while (x > 0) {
    {x ← x} [1/2] {x ← x − 1}
}
```

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  - $\rightarrow$ very successful for classical programs [Giesl et al. '16].

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- **Contribution**: Novel **modular** approach by combining expected runtimes and expected sizes.

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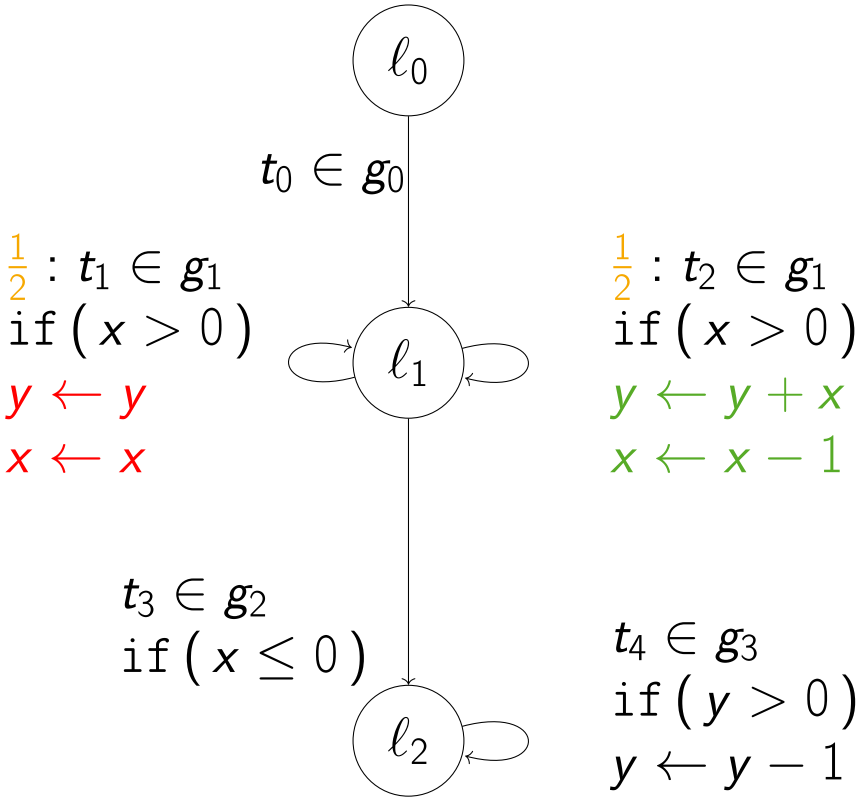

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\quad \} \\
\ell_2 : \ \texttt{while}\,(\,y > 0\,)\,\{ \\
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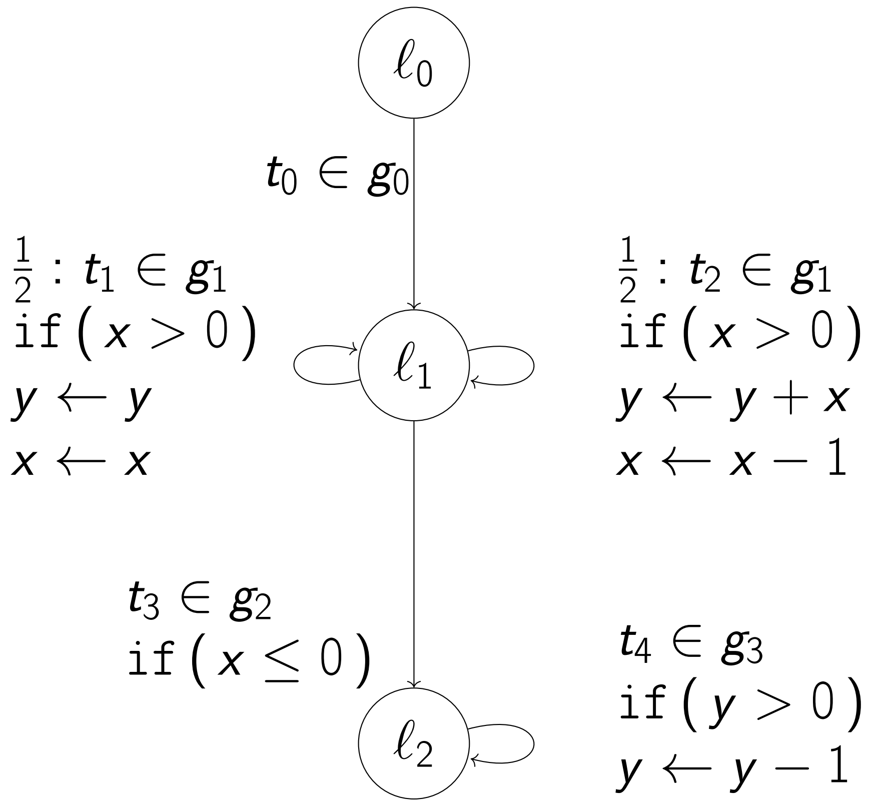
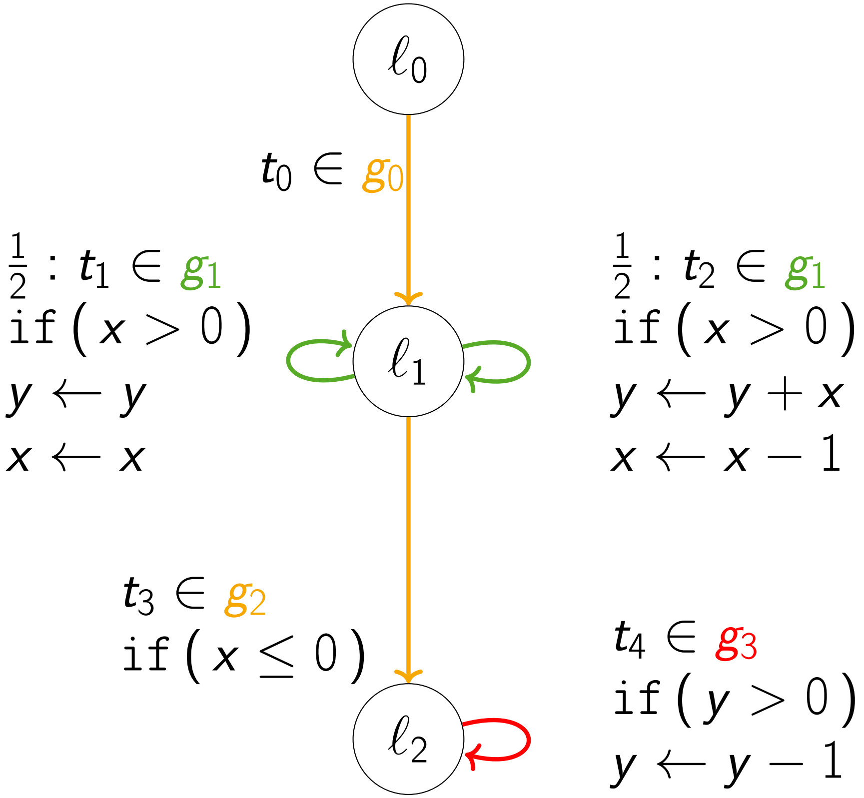
Outline

Overall Idea



Overall Idea

- For each general transition $g \in \mathcal{GT}$:



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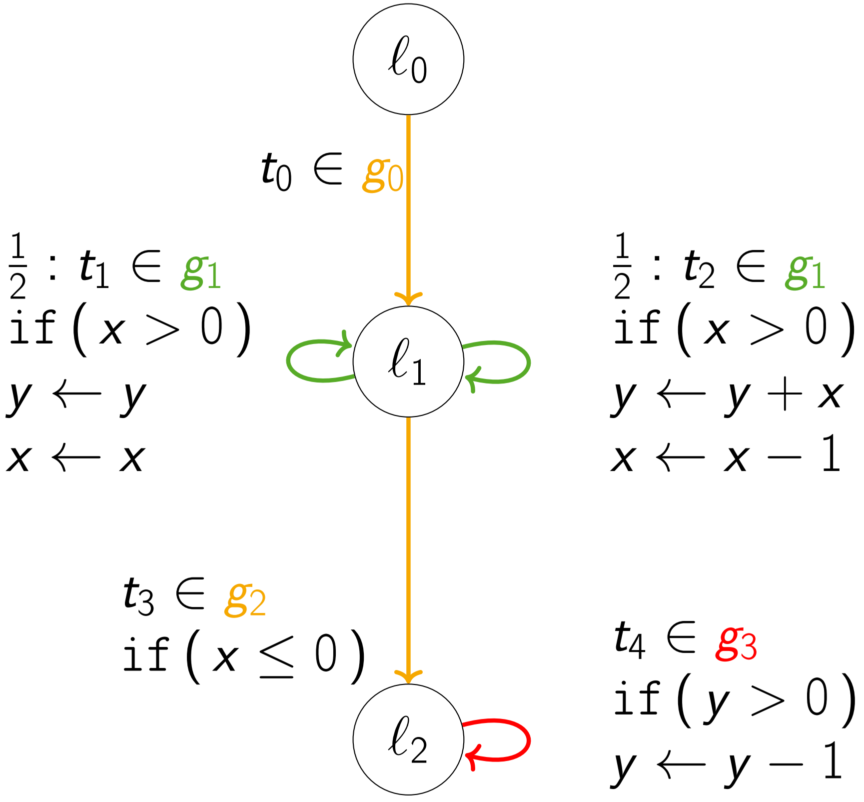
- For each general transition $g \in \mathcal{GT}$:
  $\mathcal{R}(g) =$ number of executions of g in run of full program.



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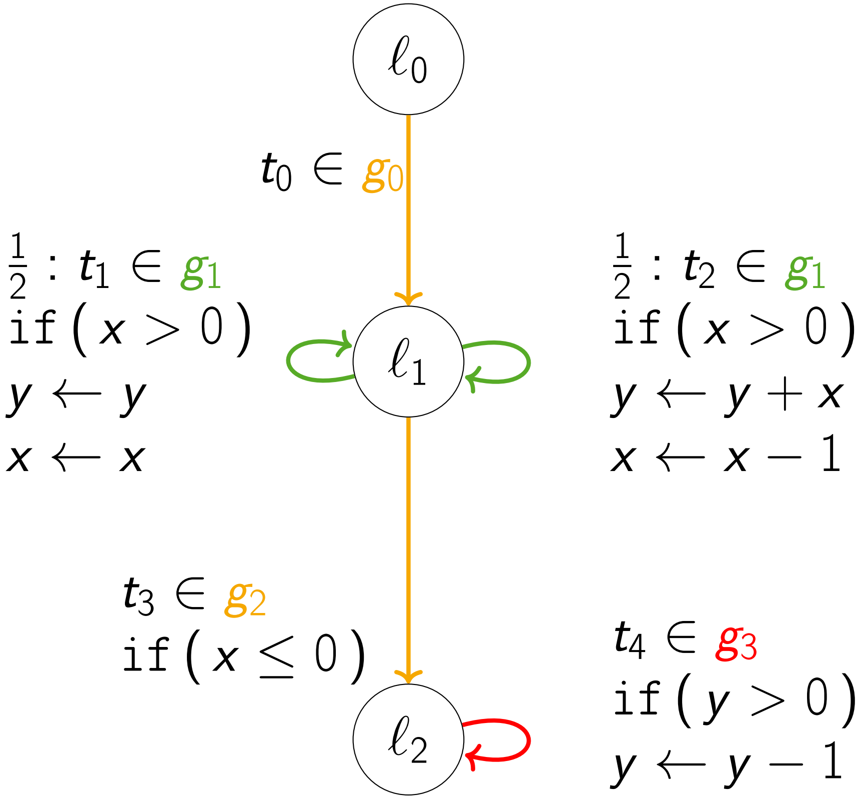

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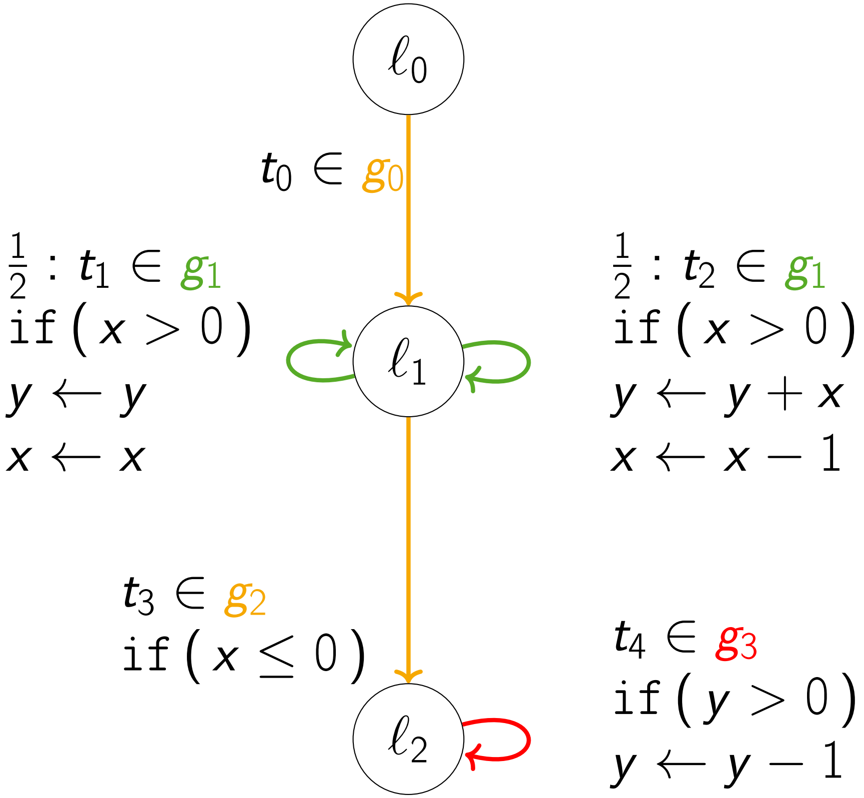

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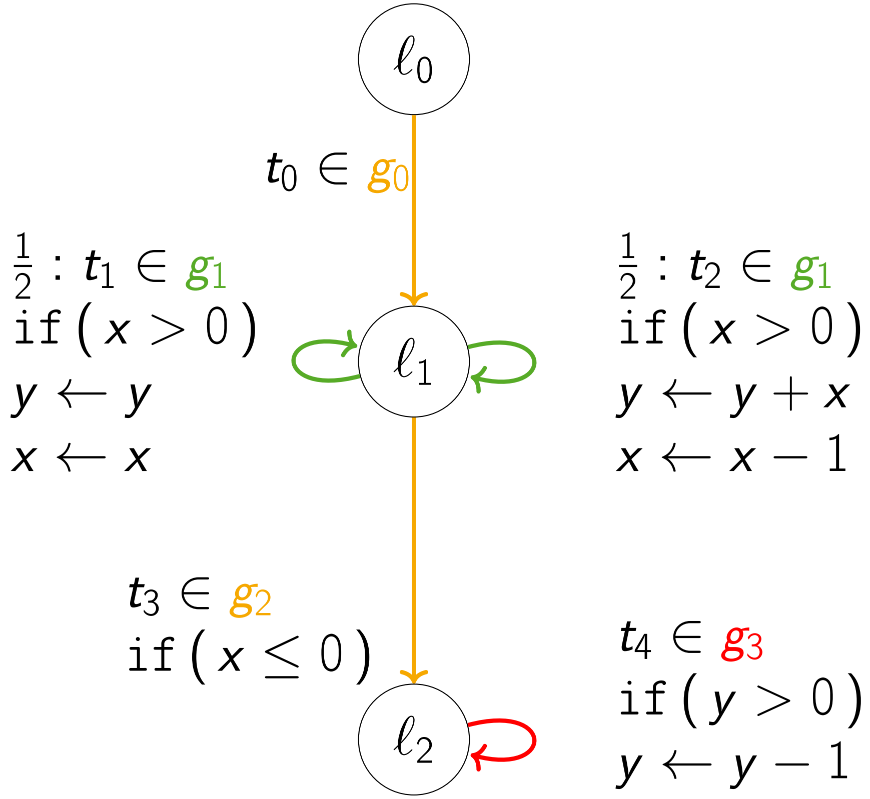
  $\rightarrow$ Contribution: **Modular** inference of bound on $\mathbb{E}(\mathcal{R}(g))$.



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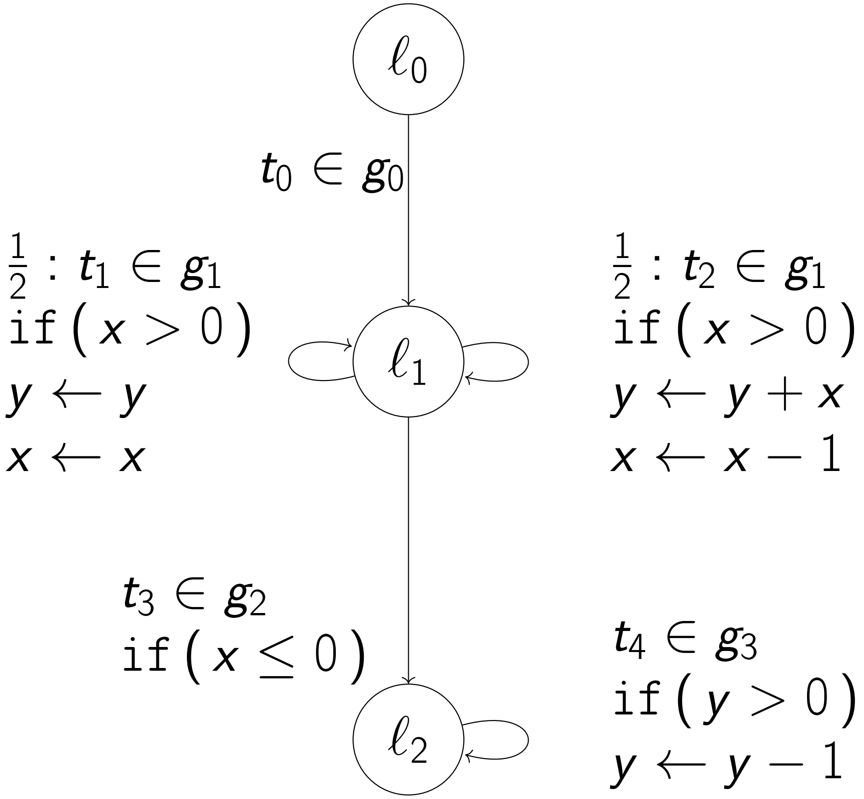

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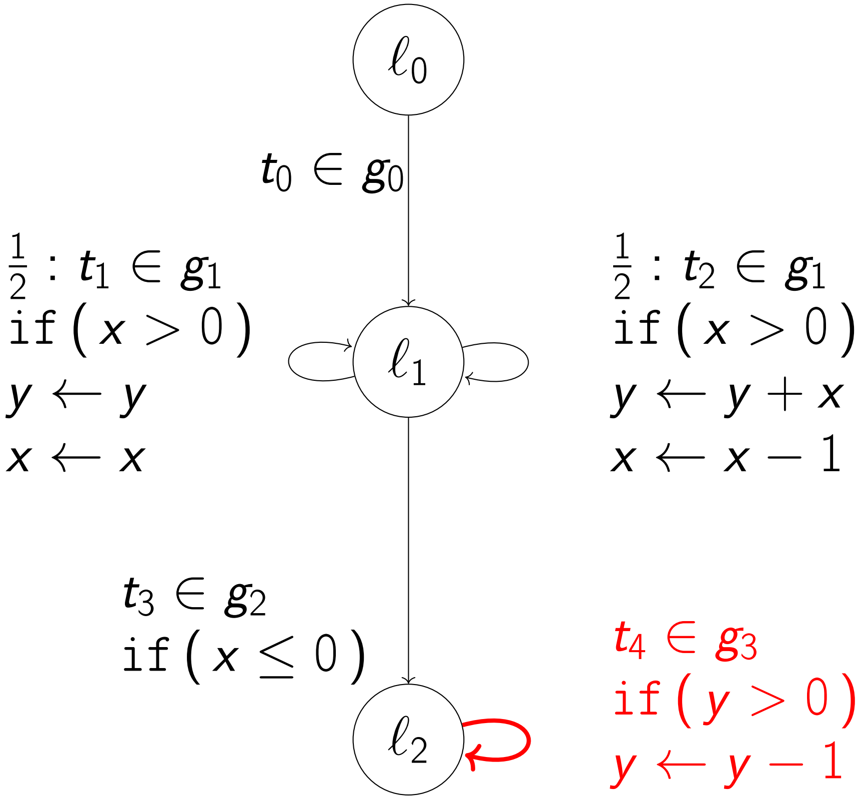

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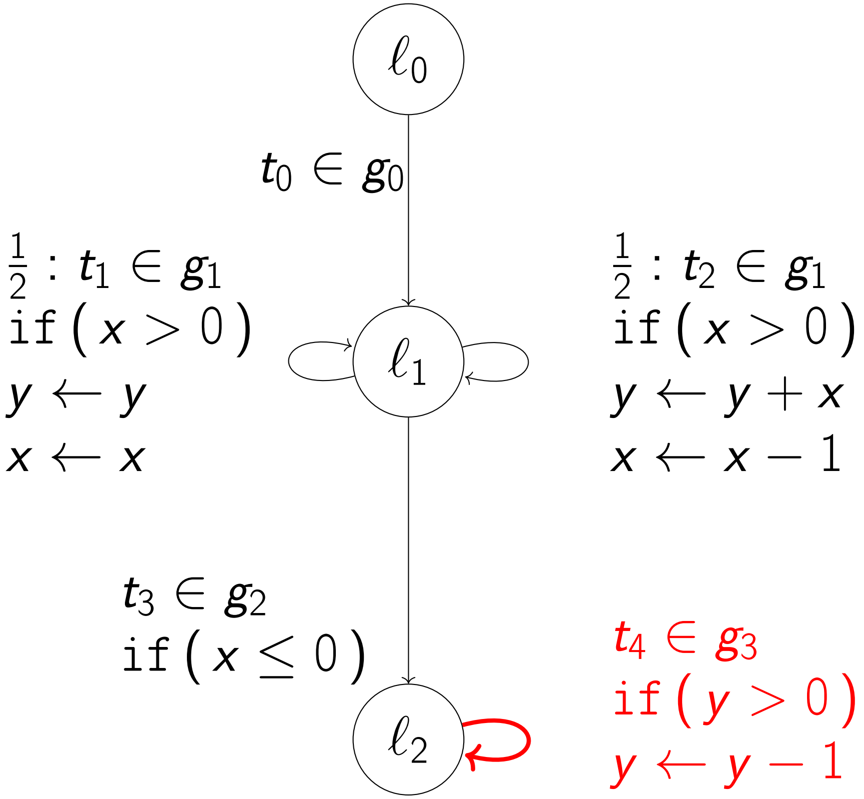
  Number of executions of $\mathcal{GT}'$ in run of full program?



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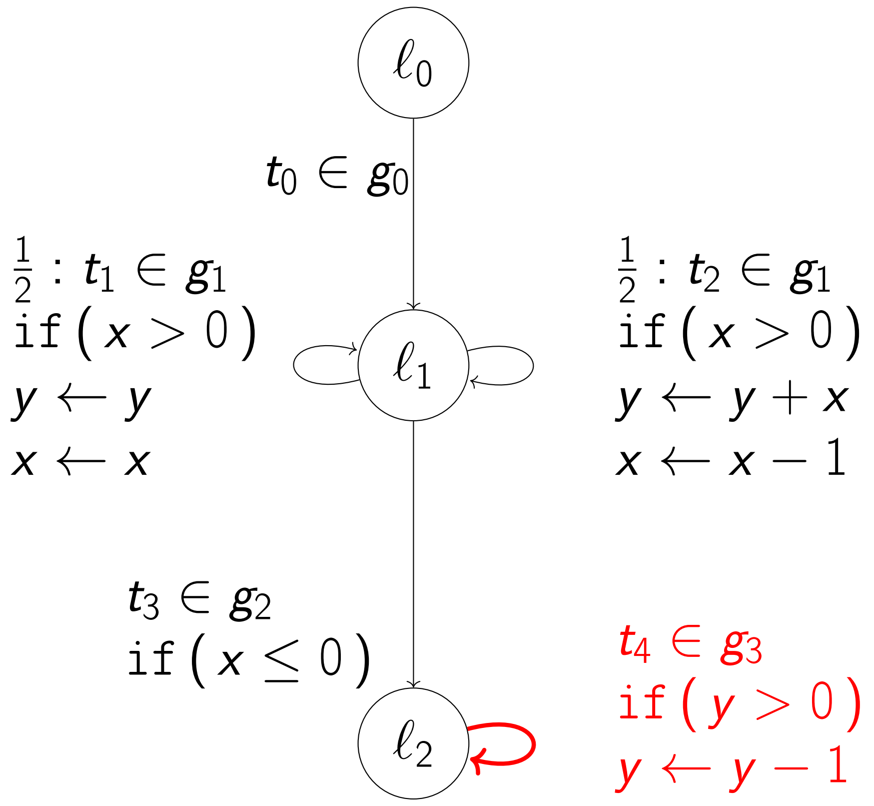
  $\leq (\#\ \text{enter}\ (\mathcal{GT}')) \cdot (\text{local-runtime}(\mathcal{GT}')\,[v/\text{``size''}_{\text{enter}\mathcal{GT}'}(v)])$



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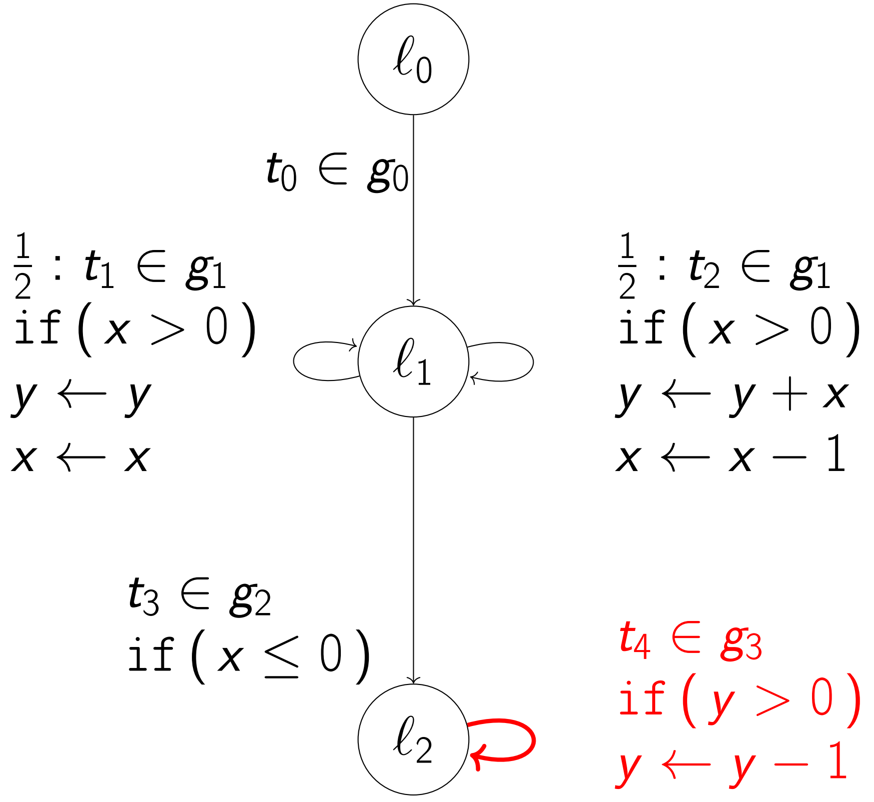

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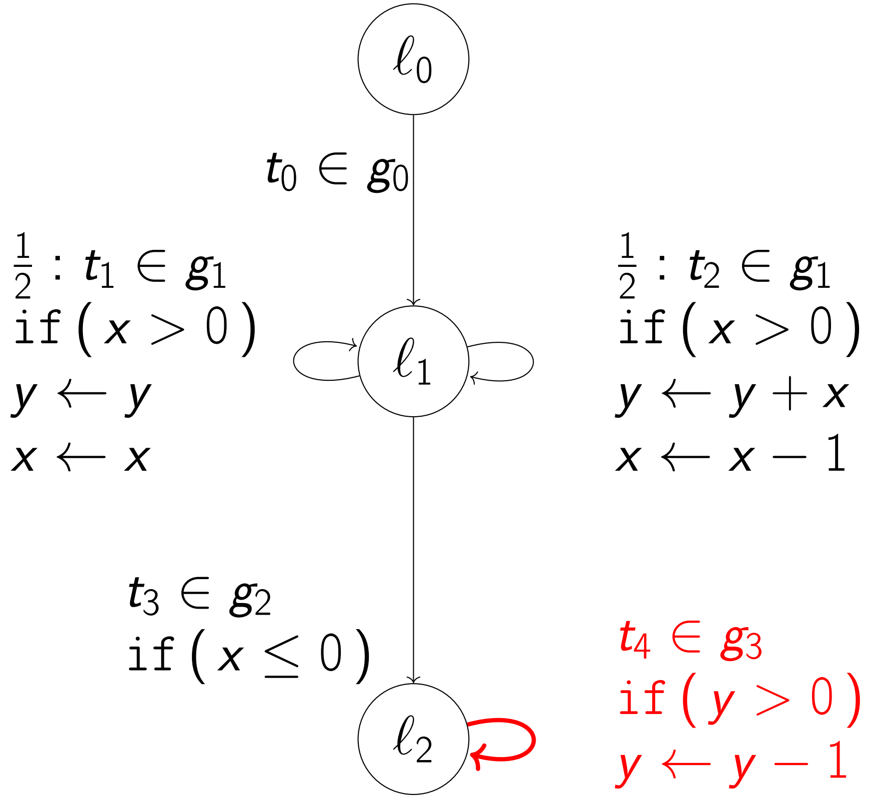

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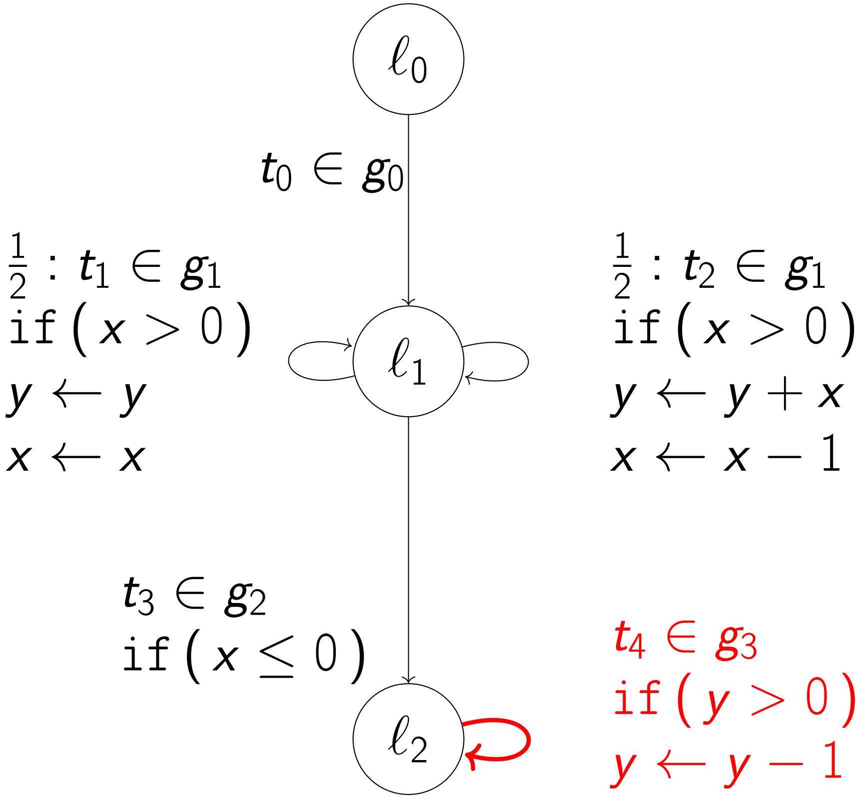

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Computing Expected Runtime Bounds

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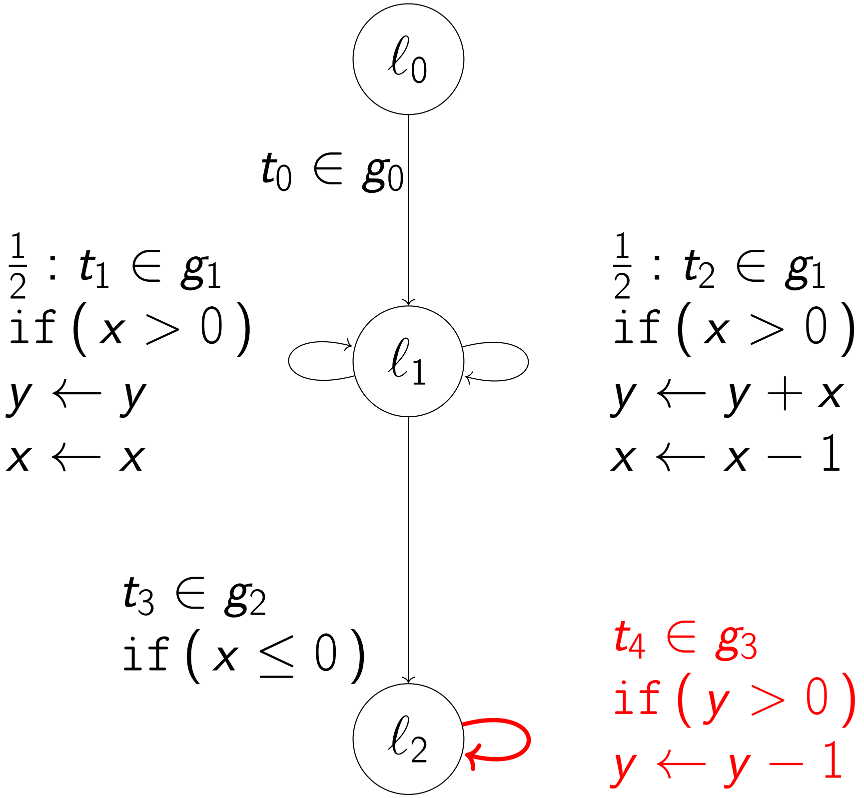
  $\rightarrow$ Expected value is not multiplicative.



Computation

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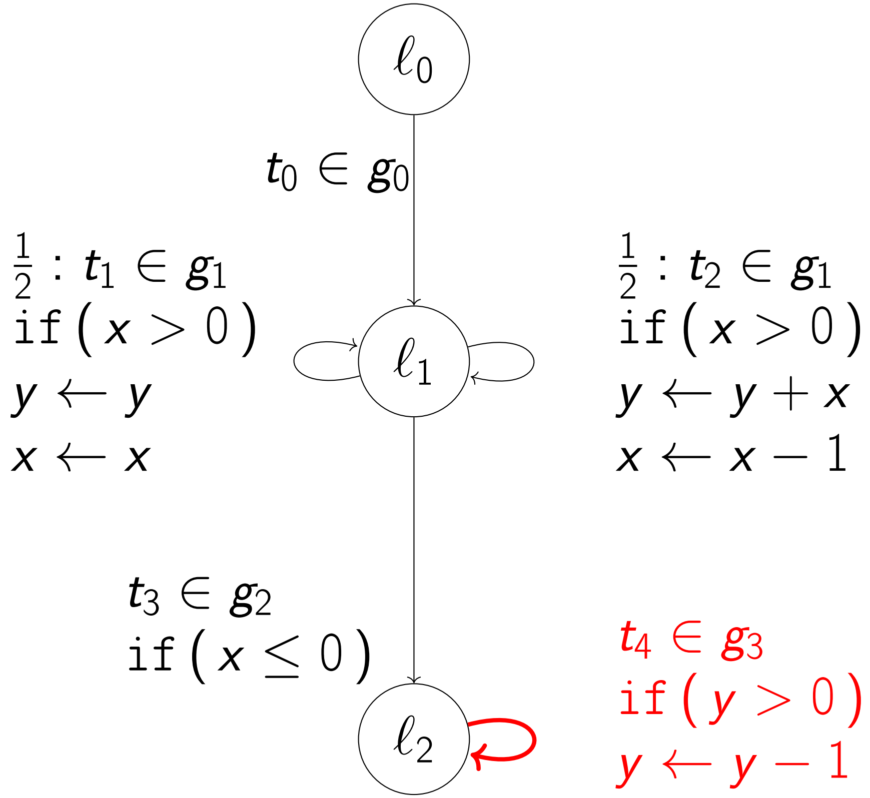

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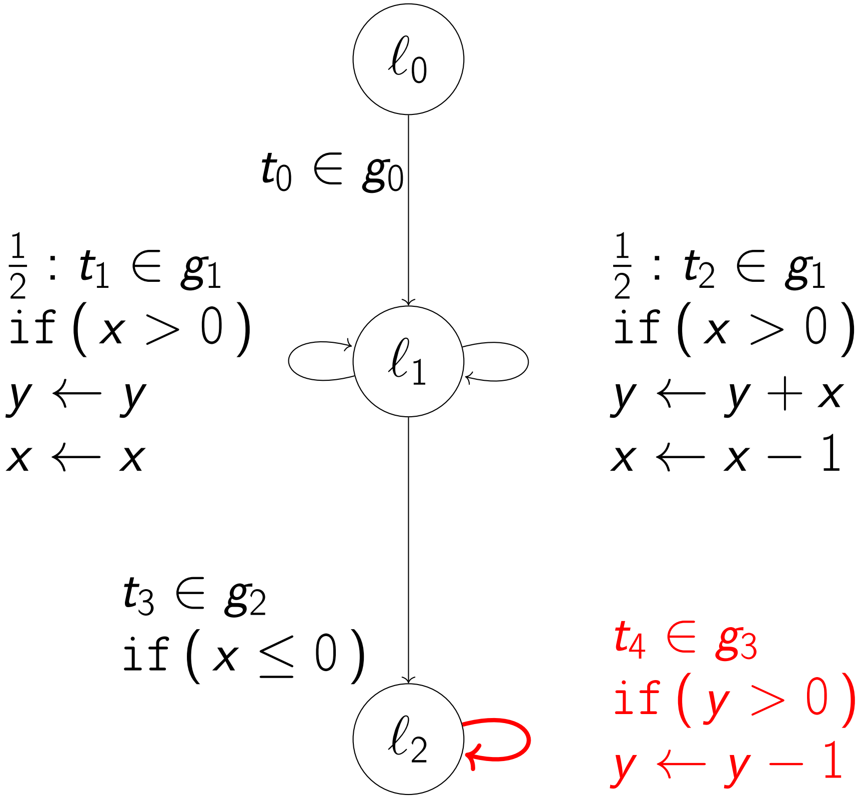

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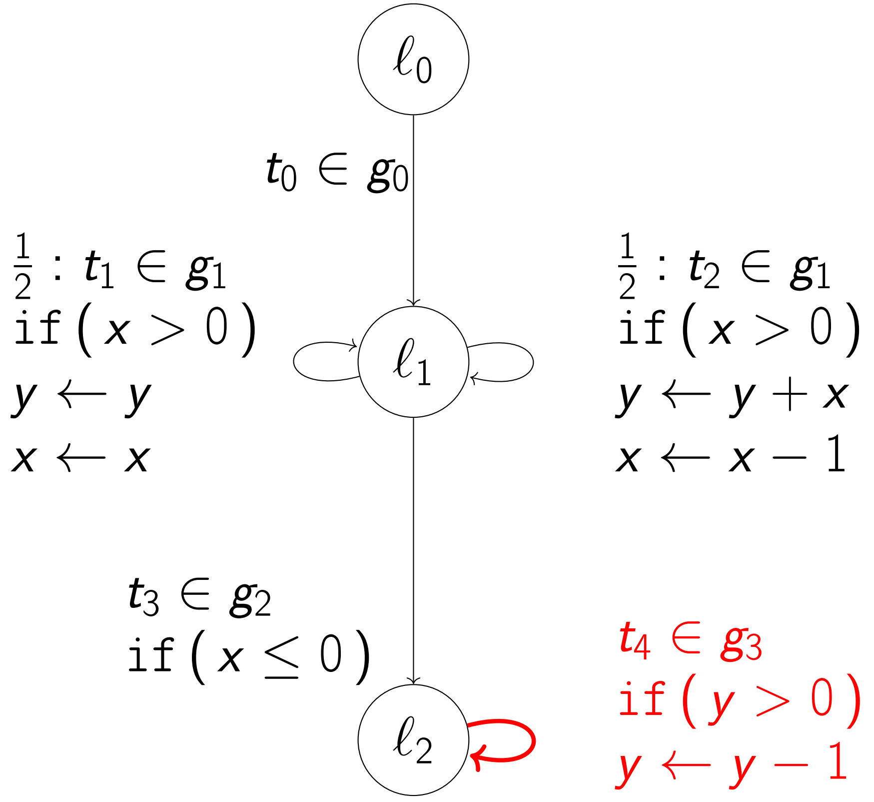

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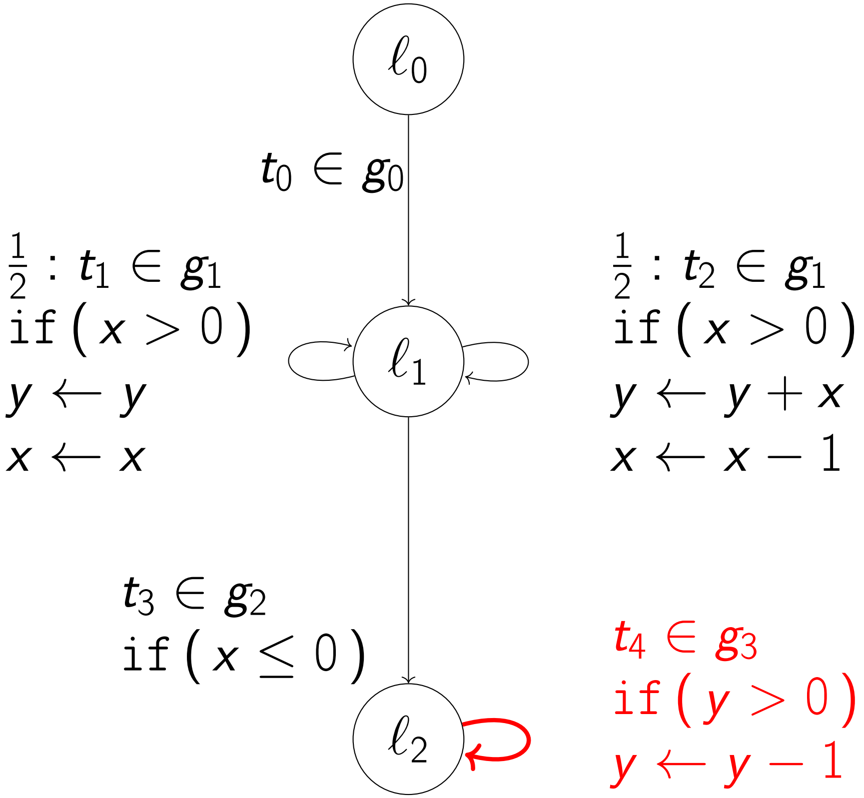

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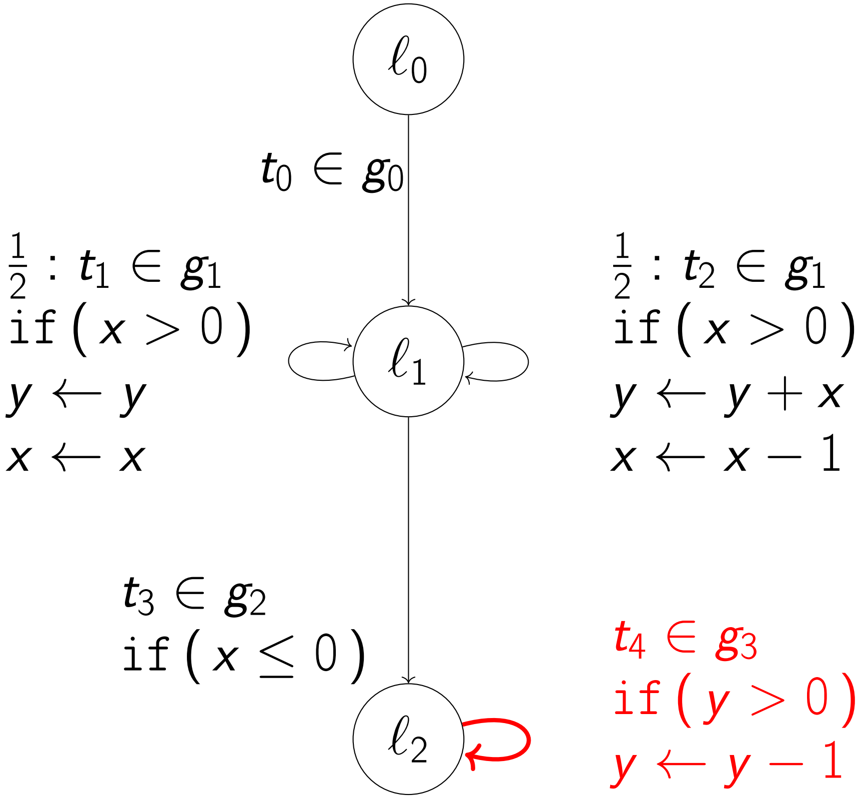
  $\#$ enter $(\mathcal{GT}')$:

  $\rightarrow$ Use (classical) worst-case bound from [Giesl et al. '16].



Computing Expected Runtime Bounds

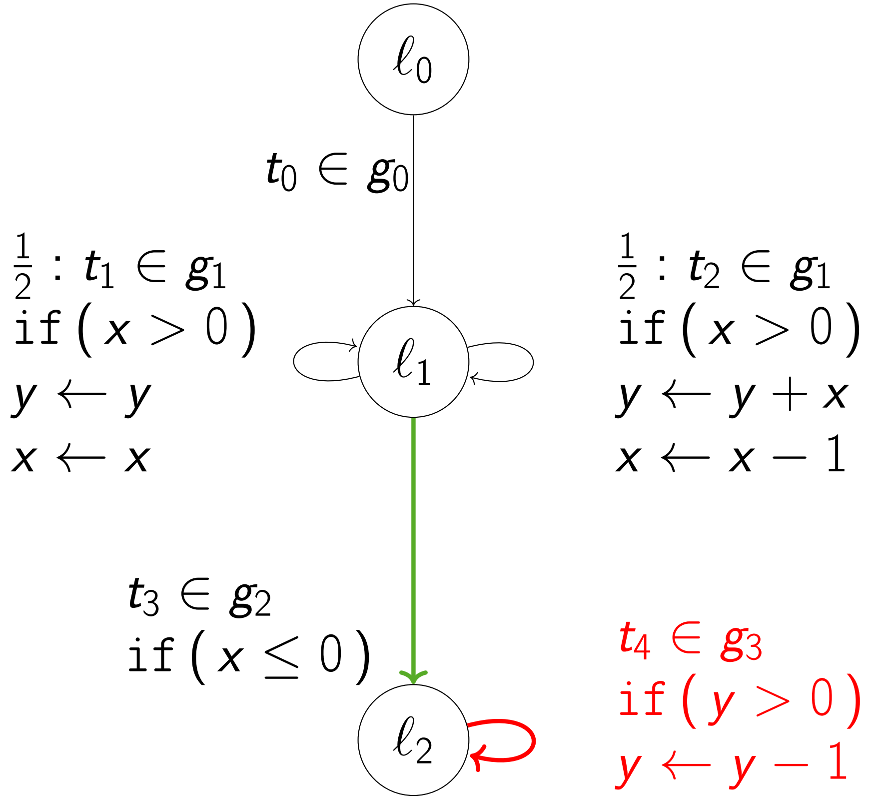
Computation

$$(\#\,\text{enter}\,(\mathcal{GT}'))\cdot\mathbb{E}\Big(\big(\text{local-runtime}(\mathcal{GT}')\,[v/\text{``size''}_{\text{enter}\mathcal{GT}'}(v)]\big)\Big)$$

- How to over-approximate this expected value?

  $\#$ enter $(\mathcal{GT}')$:

  $\rightarrow$ Use (classical) worst-case bound from [Giesl et al. '16].



Computation

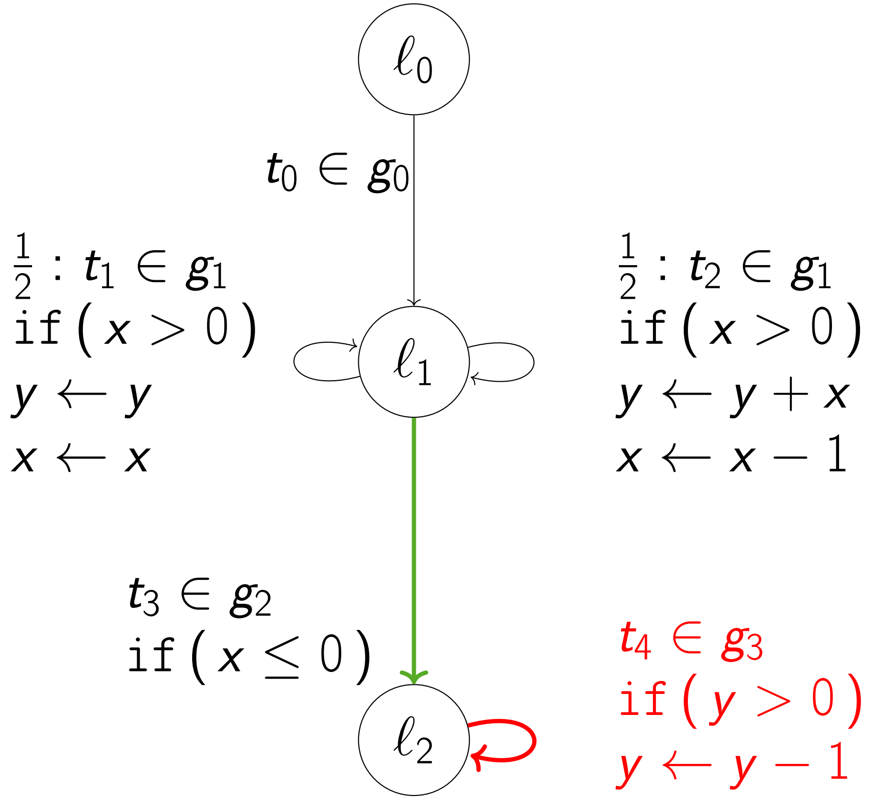
$$(\#\ \text{enter}\ (\mathcal{GT}')) \cdot \mathbb{E}\Big((\text{local-runtime}(\mathcal{GT}')\,[v/\text{“size”}_{\text{enter}\,\mathcal{GT}'}(v)])\Big)$$

- How to over-approximate this expected value?

  # enter $(\mathcal{GT}')$:

  → Use (classical) worst-case bound from [Giesl et al. '16].

  → # enter $(\mathcal{GT}') = 1$.

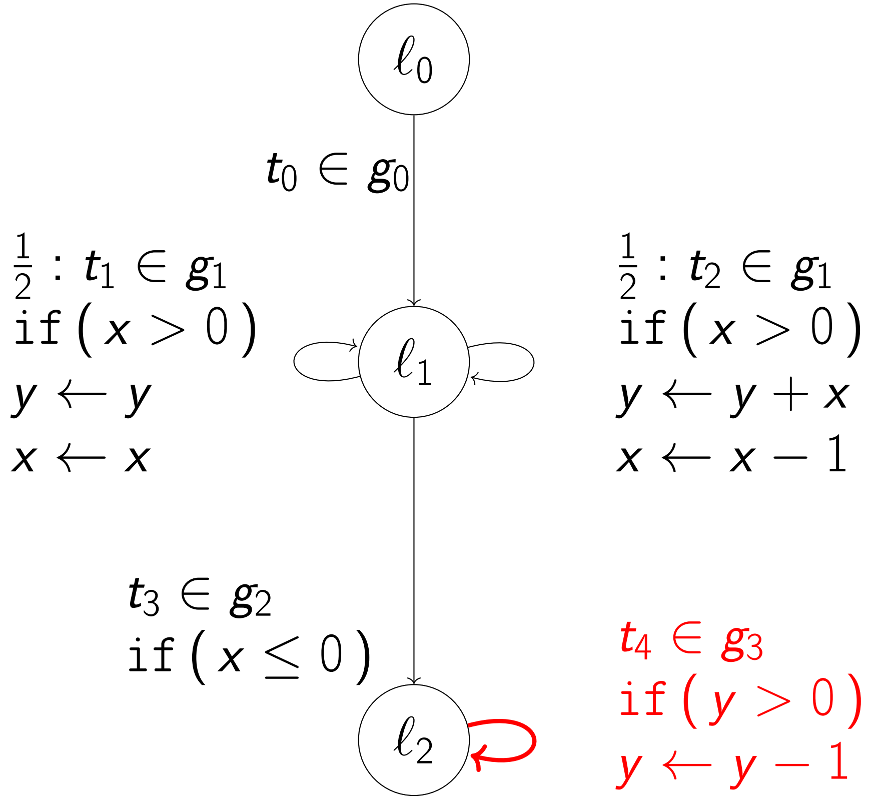


Computing Expected Runtime Bounds

Computation

$$\mathbb{E}\Big(\big(\text{local-runtime}(\mathcal{GT}')\,[v/\text{``size''}_{\text{enter}_{\mathcal{GT}'}}(v)]\big)\Big)$$

- How to over-approximate this expected value?

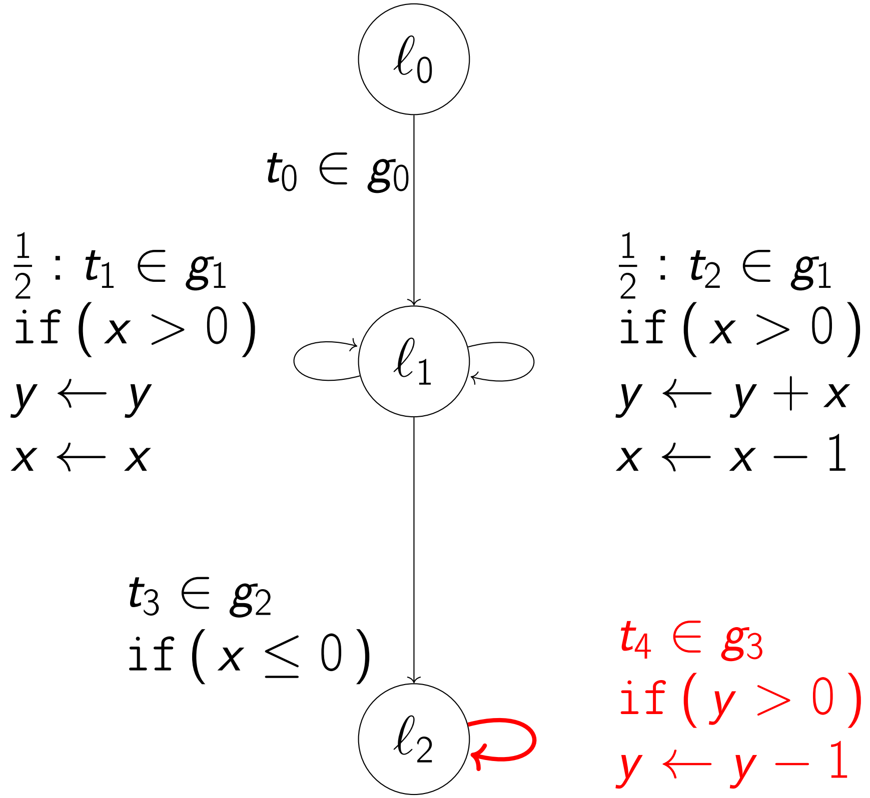


Computing Expected Runtime Bounds

Computation

$$\mathbb{E}\Big(\big(\text{local-runtime}(\mathcal{GT}')\,[v/\text{``size''}_{\text{enter}_{\mathcal{GT}'}}(v)]\big)\Big)$$

- How to over-approximate this expected value?
  - $\rightarrow$ Use linear (probabilistic) ranking function $\mathfrak{r}$ for $\mathcal{GT}'$.

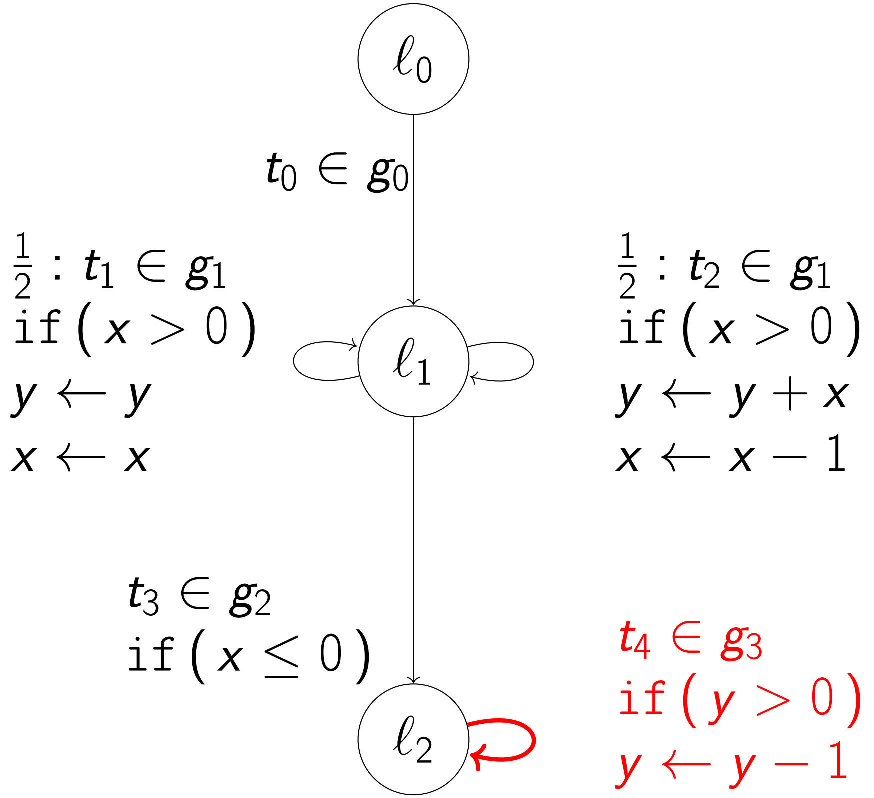


Computing Expected Runtime Bounds

Computation

$$\mathfrak{r}\left[v/\mathbb{E}(\text{“size”}_{\text{enter}_{\mathcal{GT}'}}(v))\right]$$

- How to over-approximate this expected value?
  - $\rightarrow$ Use linear (probabilistic) ranking function $\mathfrak{r}$ for $\mathcal{GT}'$.

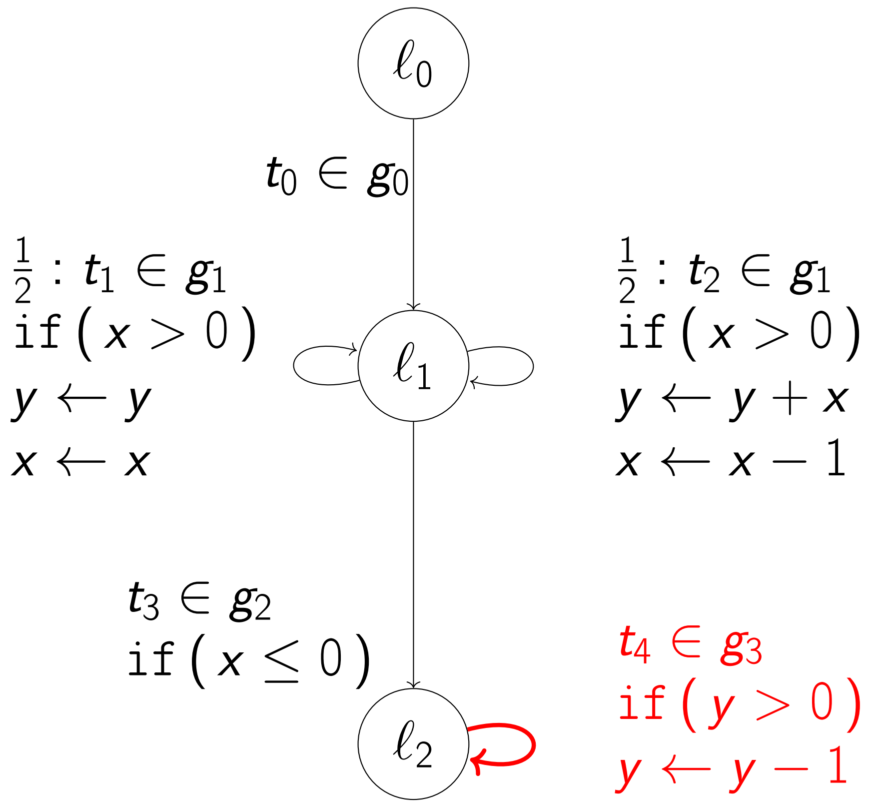


Computing Expected Runtime Bounds

Computation

$$\mathfrak{r}\left[v/\mathbb{E}(\text{``size''}_{\text{enter}\mathcal{GT}'}(v))\right]$$

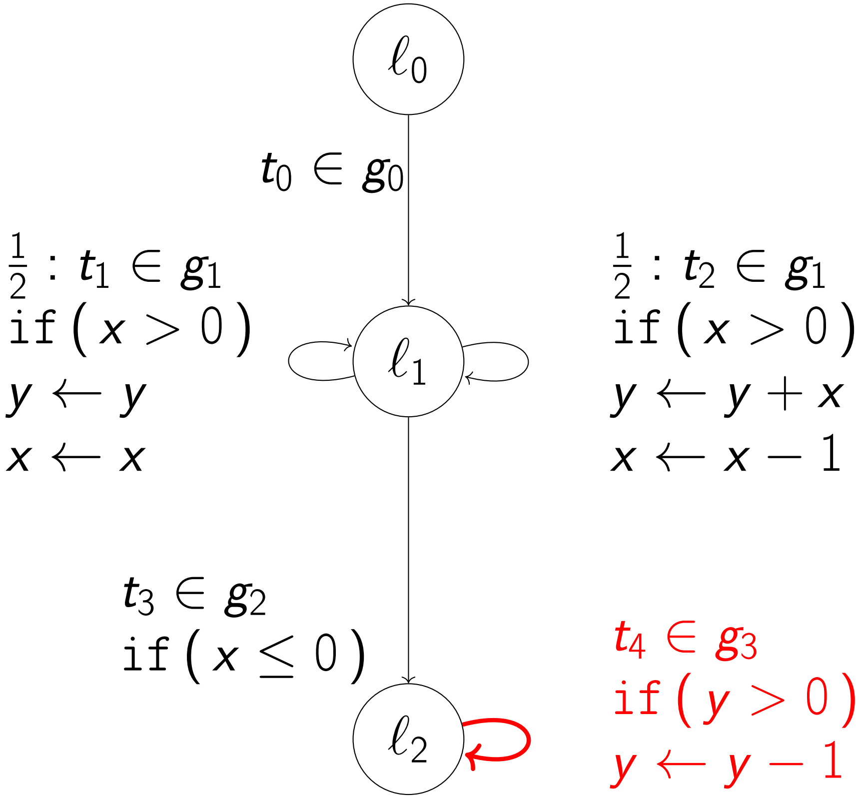
- How to over-approximate this expected value?
  - $\rightarrow$ Use linear (probabilistic) ranking function $\mathfrak{r}$ for $\mathcal{GT}'$.
  - $\rightarrow$ Already seen: y is a ranking function for $\mathcal{GT}'$.



Computation

$$y\,\left[y/\mathbb{E}(\text{``size''}_{\text{enter}\mathcal{GT}'}(y))\right]$$

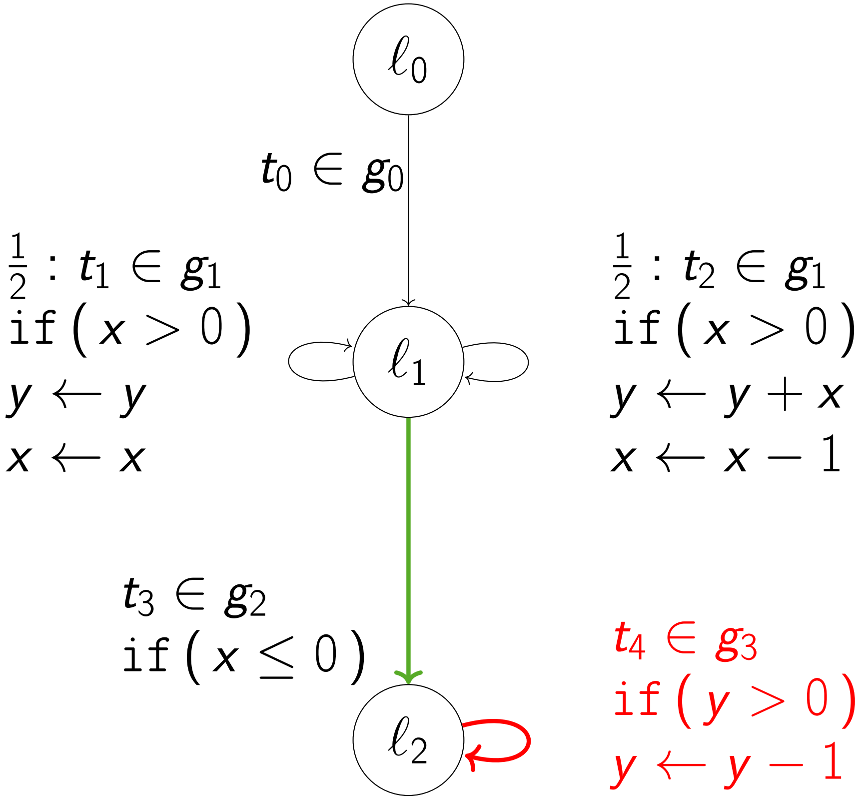
- How to over-approximate this expected value?
  - $\rightarrow$ Use linear (probabilistic) ranking function $\mathfrak{r}$ for $\mathcal{GT}'$.
  - $\rightarrow$ Already seen: y is a ranking function for $\mathcal{GT}'$.



Computation

$$y\left[y/\mathbb{E}(\text{"size"}_{\text{enter}\mathcal{GT}'}(y))\right]$$

- How to over-approximate this expected value?
  - $\to$ Use linear (probabilistic) ranking function $\mathfrak{r}$ for $\mathcal{GT}'$.
  - $\to$ Already seen: y is a ranking function for $\mathcal{GT}'$.
  - $\to$ Later: expected size of y after g_2: $x_0^2 + y_0$.

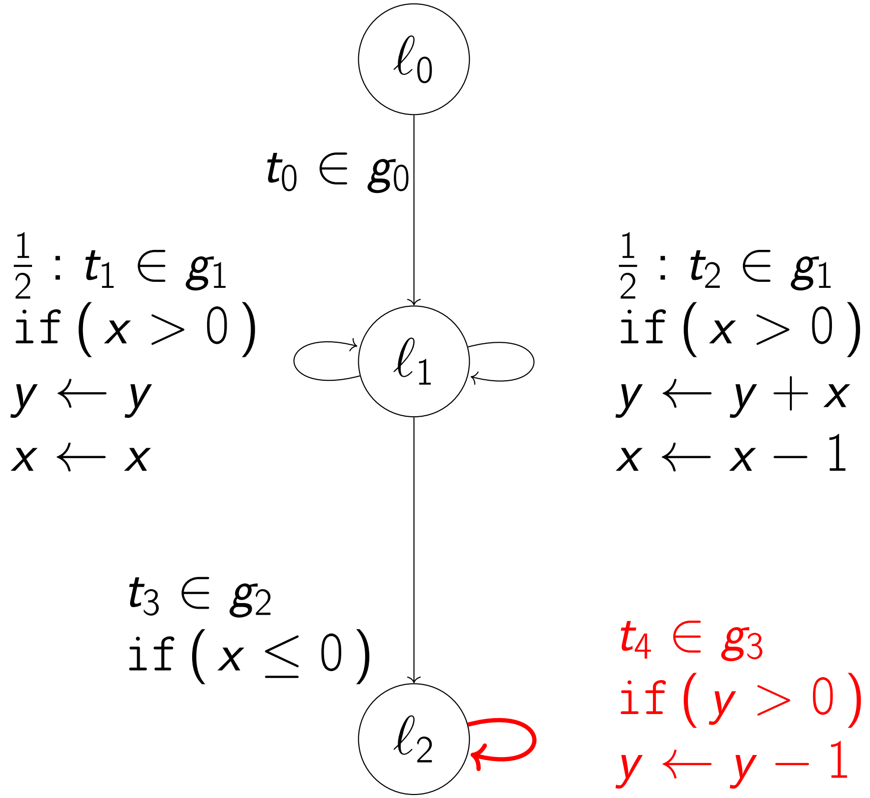


Computing Expected Runtime Bounds

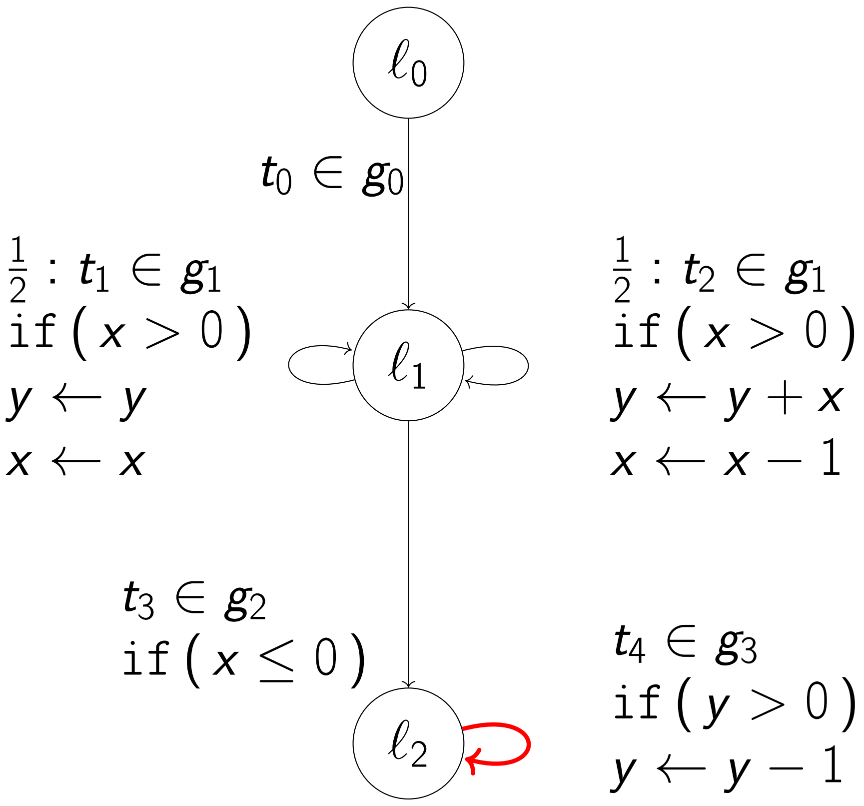
Computation

$x_0^2 + y_0$

- How to over-approximate this expected value?
  - $\rightarrow$ Use linear (probabilistic) ranking function $\mathfrak{r}$ for $\mathcal{GT}'$.
  - $\rightarrow$ Already seen: y is a ranking function for $\mathcal{GT}'$.
  - $\rightarrow$ Later: expected size of y after g_2: $x_0^2 + y_0$.



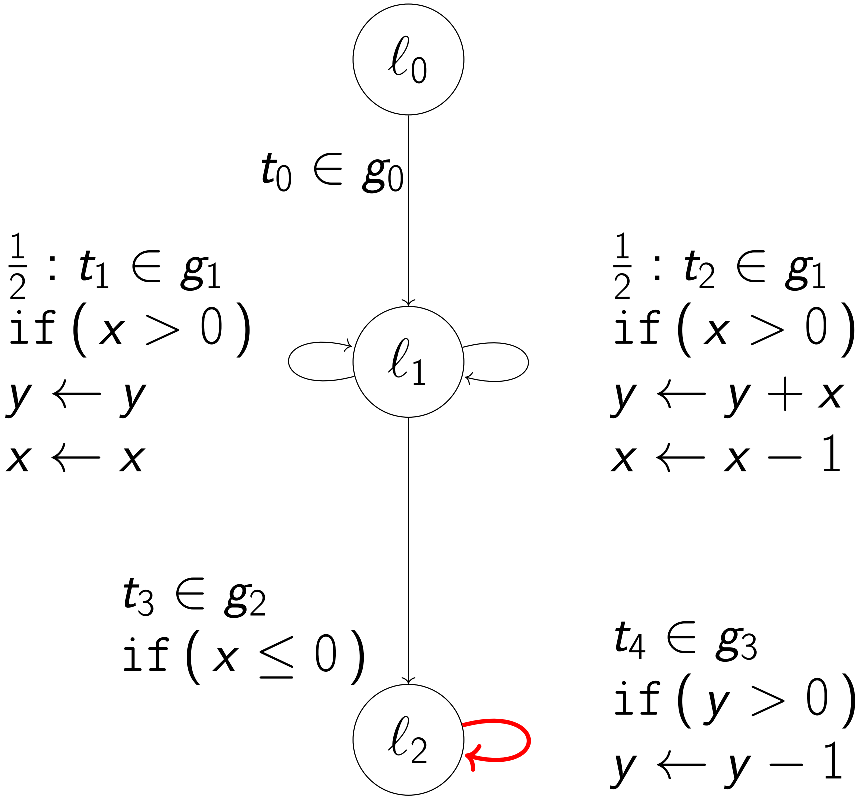
Summary


ℓ_0
$t_0 \in g_0$
$\frac{1}{2} : t_1 \in g_1$
if ($x > 0$)
$y \leftarrow y$
$x \leftarrow x$
ℓ_1
$\frac{1}{2} : t_2 \in g_1$
if ($x > 0$)
$y \leftarrow y + x$
$x \leftarrow x - 1$
$t_3 \in g_2$
if ($x \leq 0$)
ℓ_2
$t_4 \in g_3$
if ($y > 0$)
$y \leftarrow y - 1$

Computing Expected Runtime Bounds

Summary

Overall expected runtime of program:

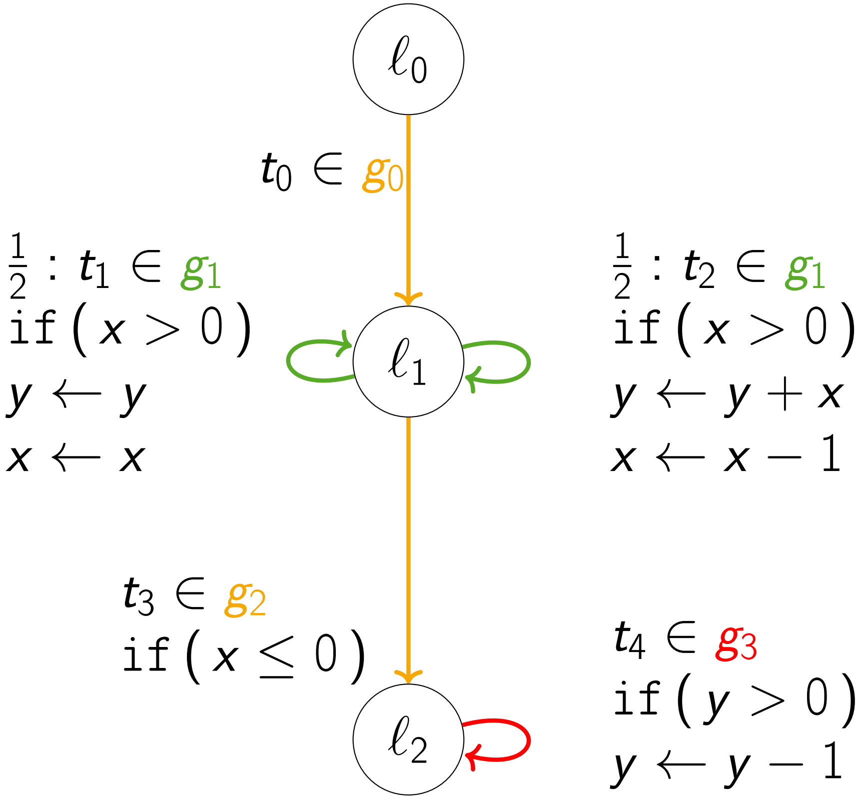


Computing Expected Runtime Bounds

Summary

Overall expected runtime of program:

$$1 + 2 \cdot x_0 + 1 + x_0^2 + y_0$$

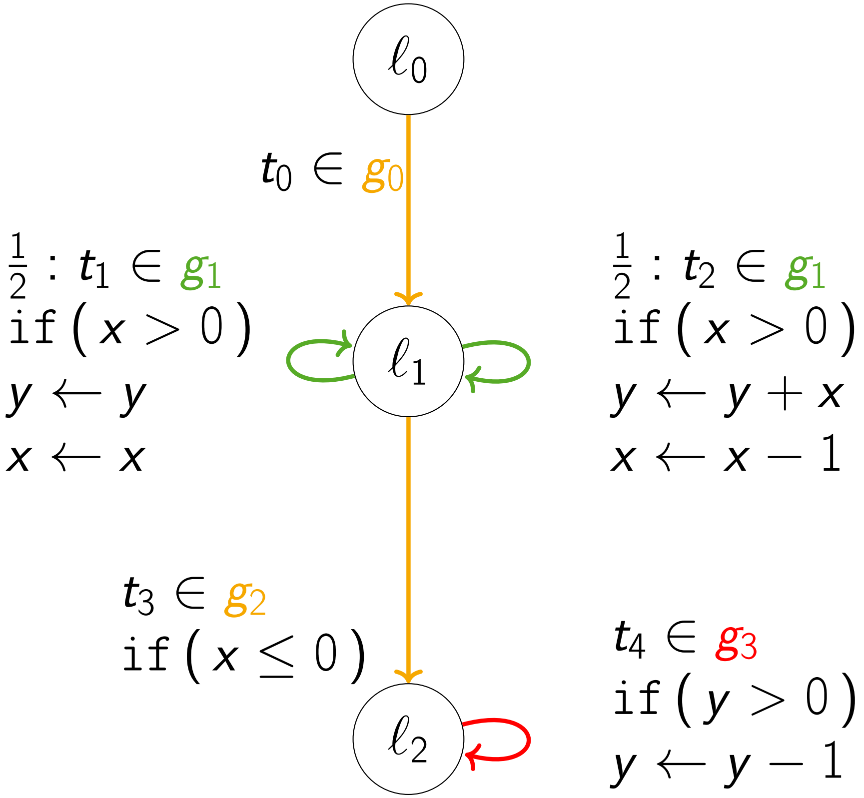


Summary

Overall expected runtime of program:

$1 + 2 \cdot x_0 + 1 + x_0^2 + y_0$

- Implementation: heuristic chooses sub-programs.



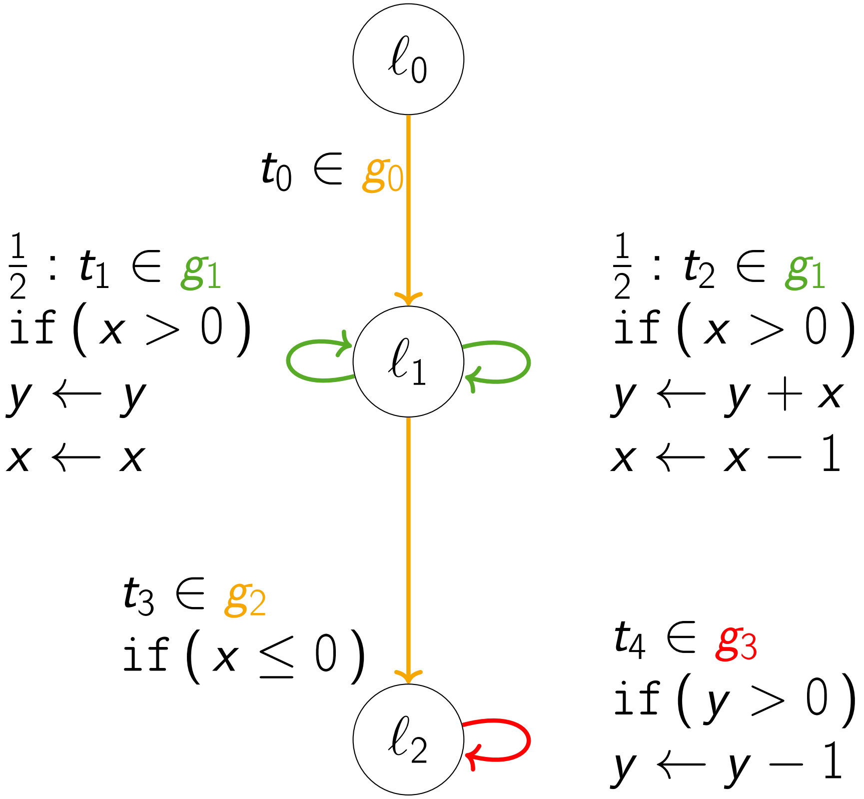
Computing Expected Runtime Bounds

Summary

Overall expected runtime of program:

$1 + 2 \cdot x_0 + 1 + x_0^2 + y_0$

- Implementation: heuristic chooses sub-programs.
- **Modular** approach for expected runtimes:



Computing Expected Runtime Bounds

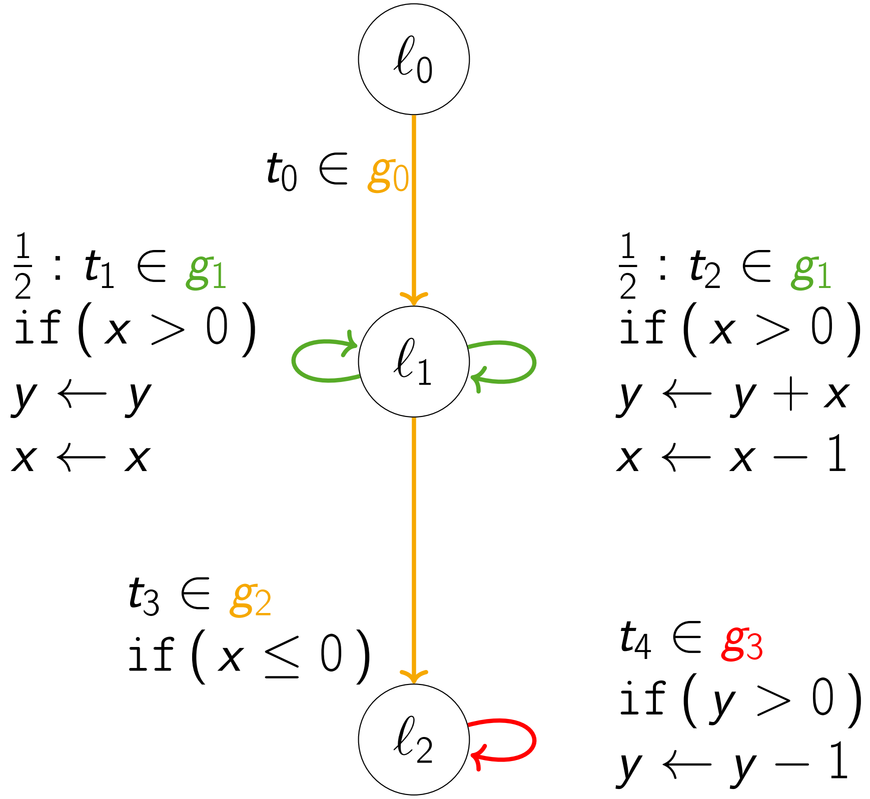
Summary

Overall expected runtime of program:

$$1 + 2 \cdot x_0 + 1 + x_0^2 + y_0$$

- Implementation: heuristic chooses sub-programs.
- **Modular** approach for expected runtimes:

  $\rightarrow$ Use of (classical) worst-case runtime bounds.



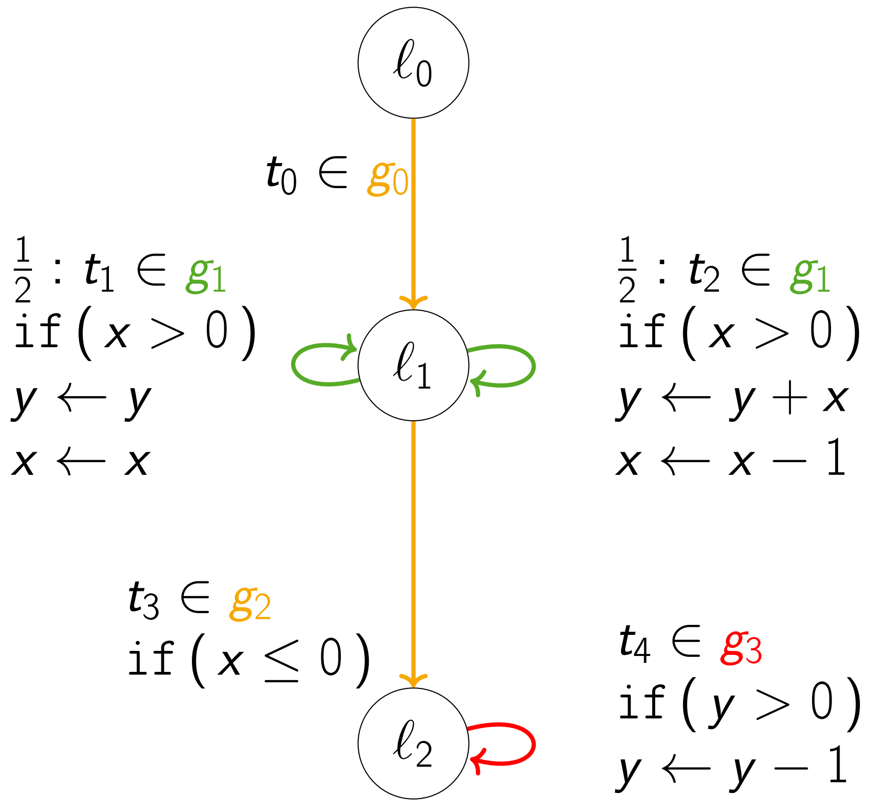
Computing Expected Runtime Bounds

Summary

Overall expected runtime of program:

$$1 + 2 \cdot x_0 + 1 + x_0^2 + y_0$$

- Implementation: heuristic chooses sub-programs.
- **Modular** approach for expected runtimes:
  - $\rightarrow$ Use of (classical) worst-case runtime bounds.
  - $\rightarrow$ Adaption of probabilistic ranking functions.

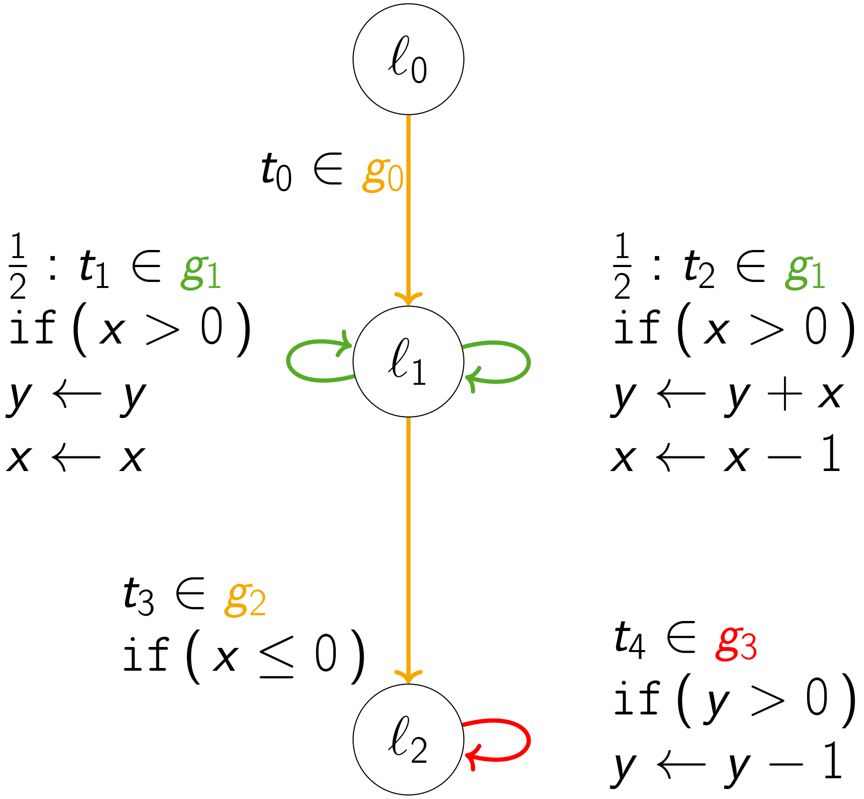


Summary

Overall expected runtime of program:

$$1 + 2 \cdot x_0 + 1 + x_0^2 + y_0$$

- Implementation: heuristic chooses sub-programs.
- **Modular** approach for expected runtimes:
  - → Use of (classical) worst-case runtime bounds.
  - → Adaption of probabilistic ranking functions.
  - → Use of expected sizes.

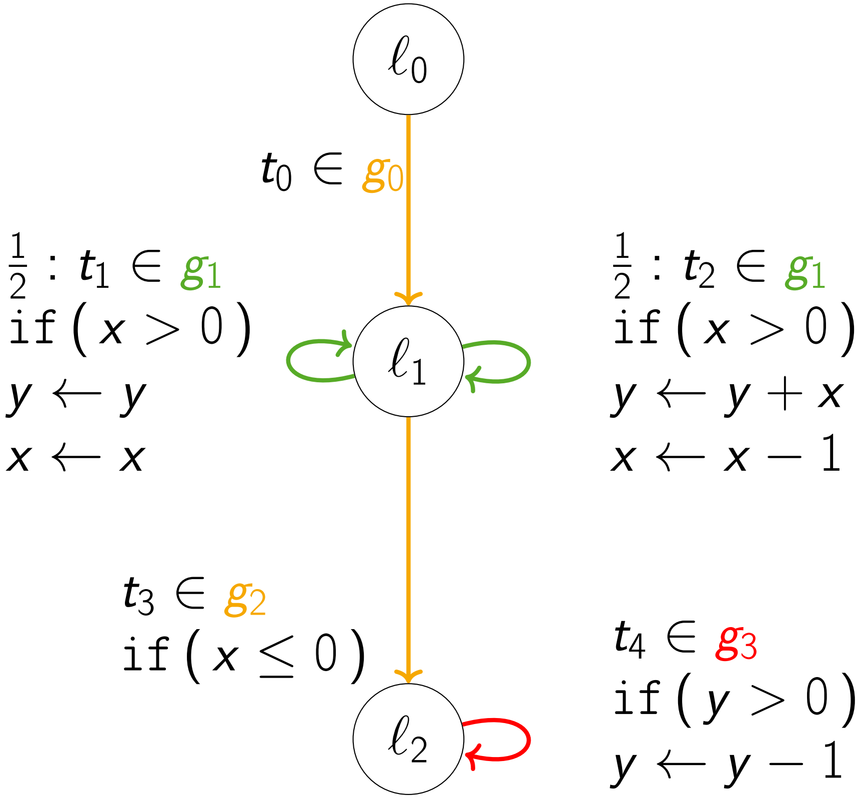


Summary

Overall expected runtime of program:

$1 + 2 \cdot x_0 + 1 + x_0^2 + y_0$

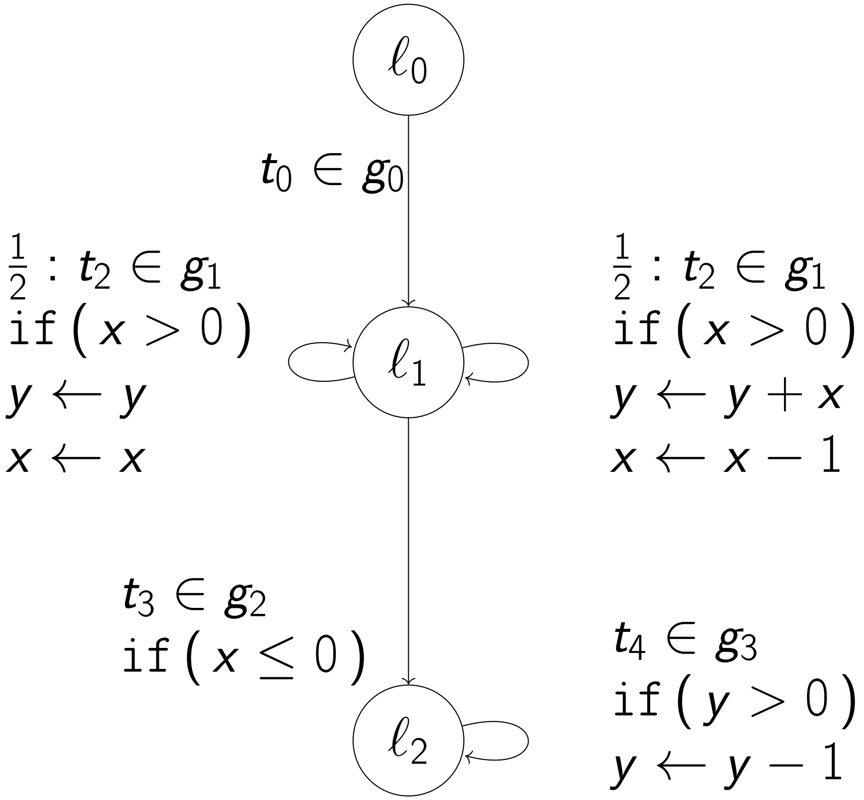
- Implementation: heuristic chooses sub-programs.
- **Modular** approach for expected runtimes:
  - → Use of (classical) worst-case runtime bounds.
  - → Adaption of probabilistic ranking functions.
  - → Use of expected sizes.
  - → Alternating computation of runtime and size bounds.



Computing Expected Size Bounds

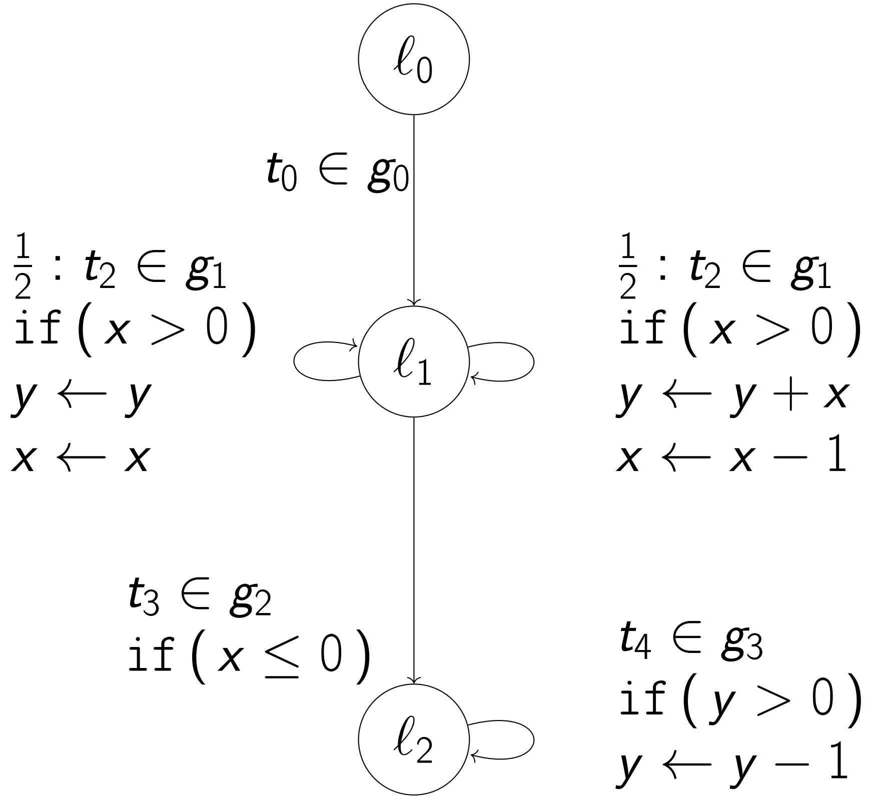
Overall Idea

Overall Idea



Overall Idea

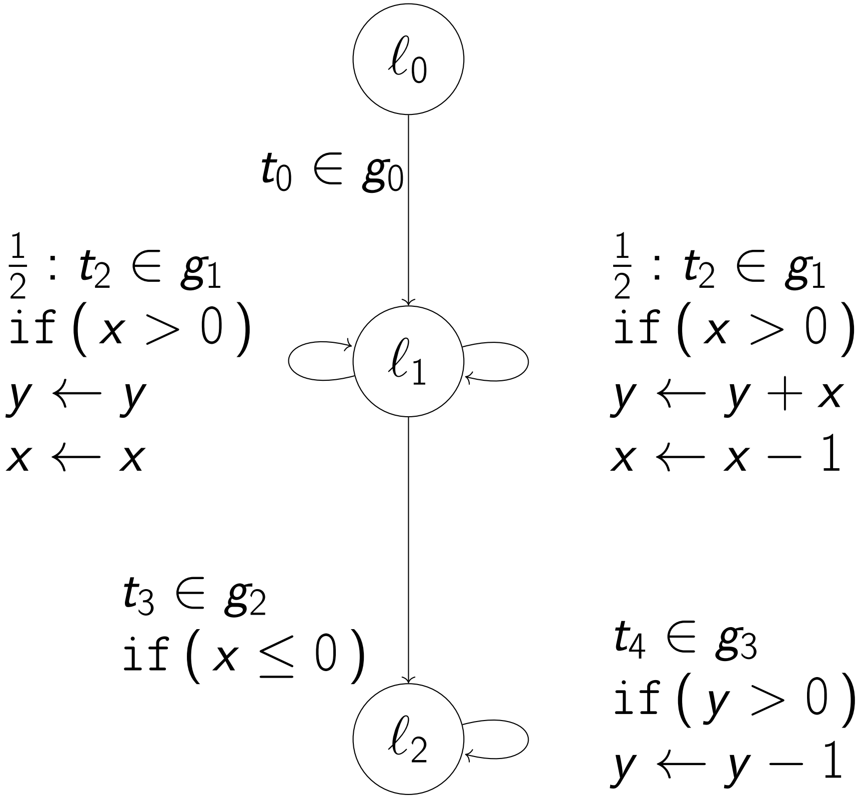
- What means "size"?



Computing Expected Size Bounds

Overall Idea

- What means "size"?
  → $\mathcal{S}(g, v)$: largest value v takes after execution of g

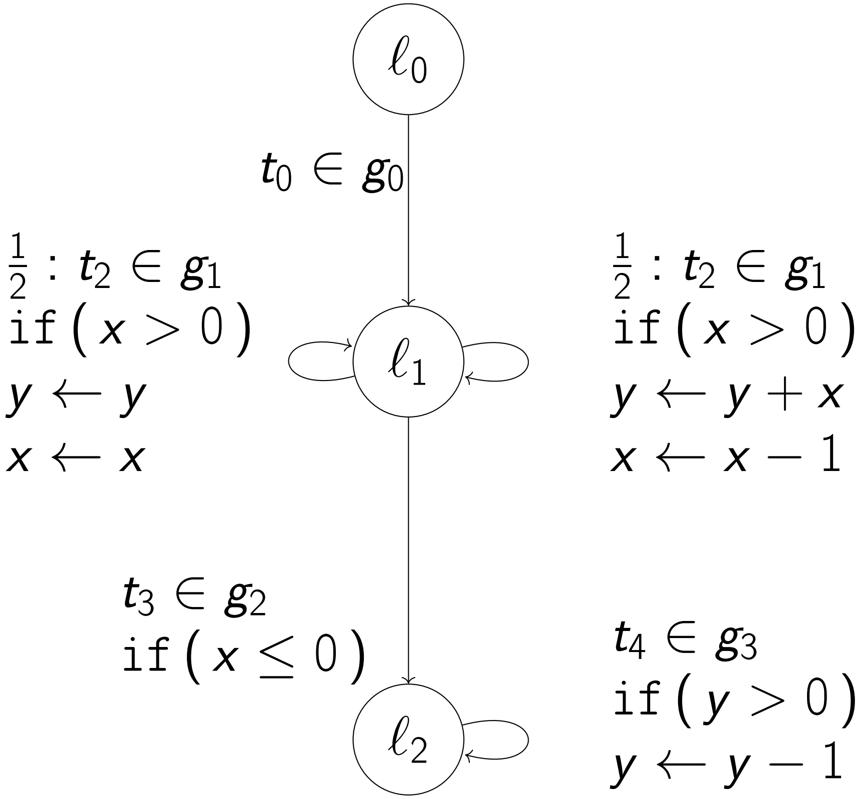


Overall Idea

- What means "size"?

  $\to \mathcal{S}(g, v)$: largest value v takes after execution of g

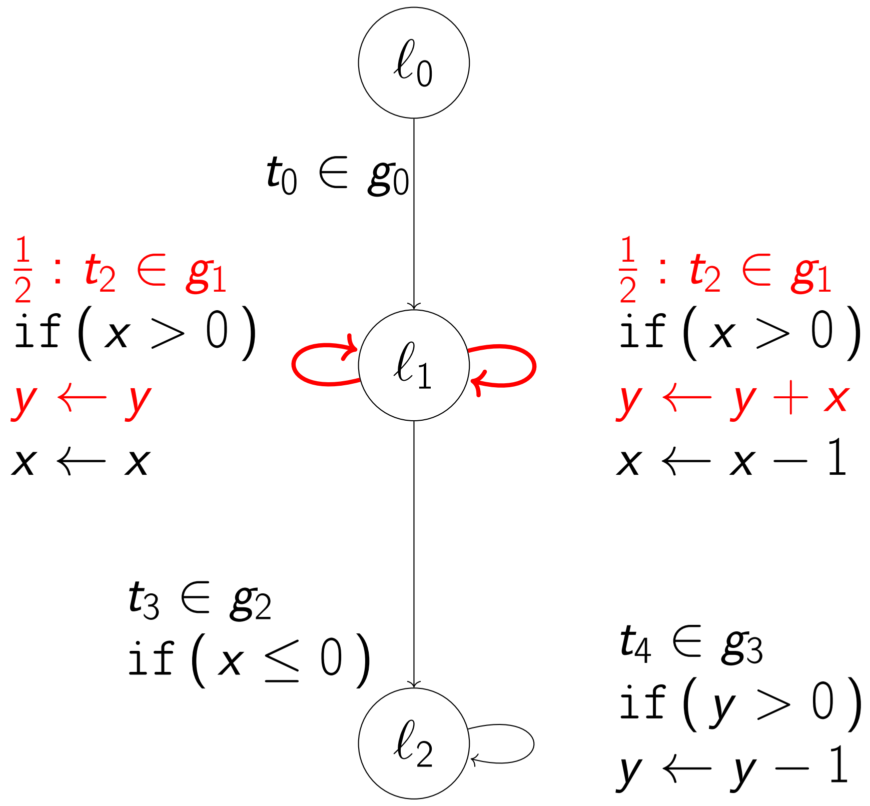
  $\leq$ incoming-size$(v) + \mathcal{R}(g) \cdot$ worst-case-change(g, v)



Computing Expected Size Bounds

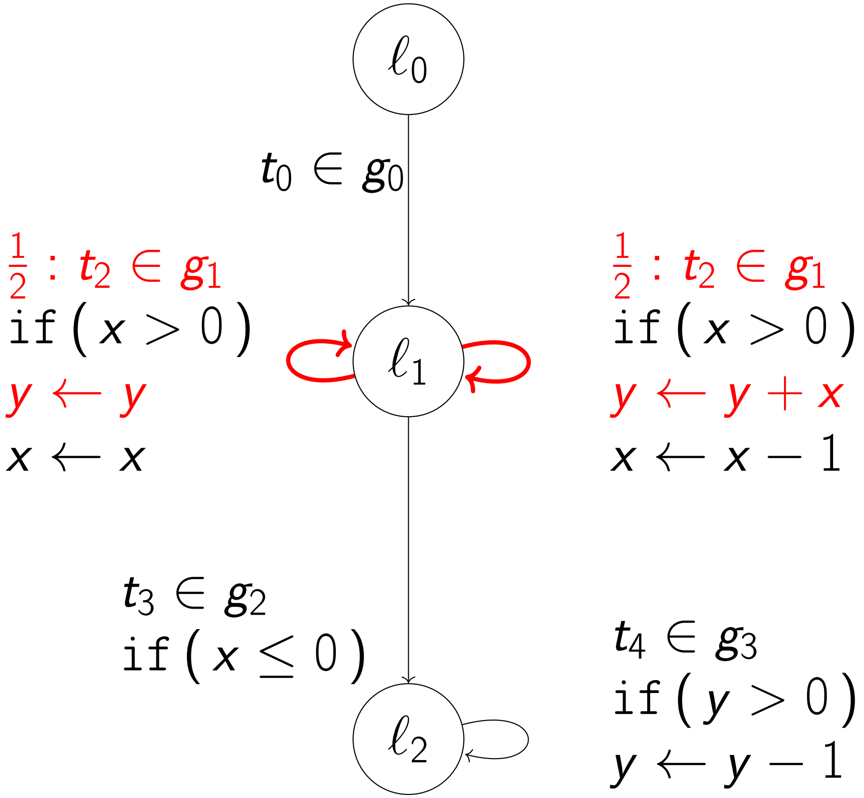
Overall Idea

- What means "size"?
  - $\to \mathcal{S}(g_1, y)$ largest value y takes after execution of g_1
  - $\leq$ incoming-size$(y) + \mathcal{R}(g_1) \cdot$ worst-case-change(g_1, y)



Overall Idea

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

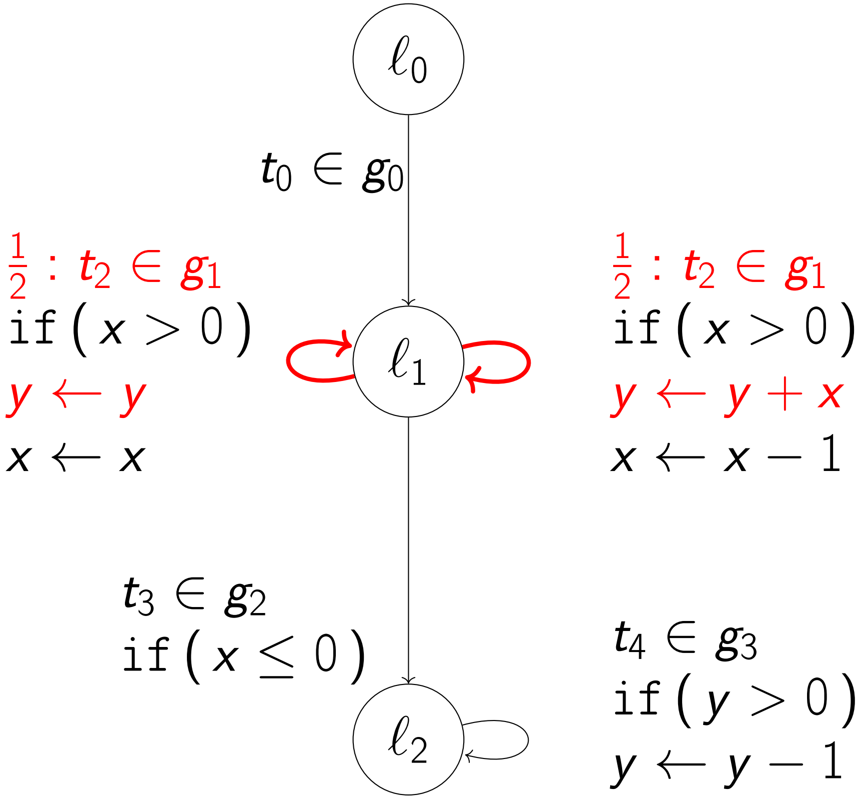


Computing Expected Size Bounds

Overall Idea

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\leq \mathbb{E}(\text{incoming-size}(y) + \mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y))$$



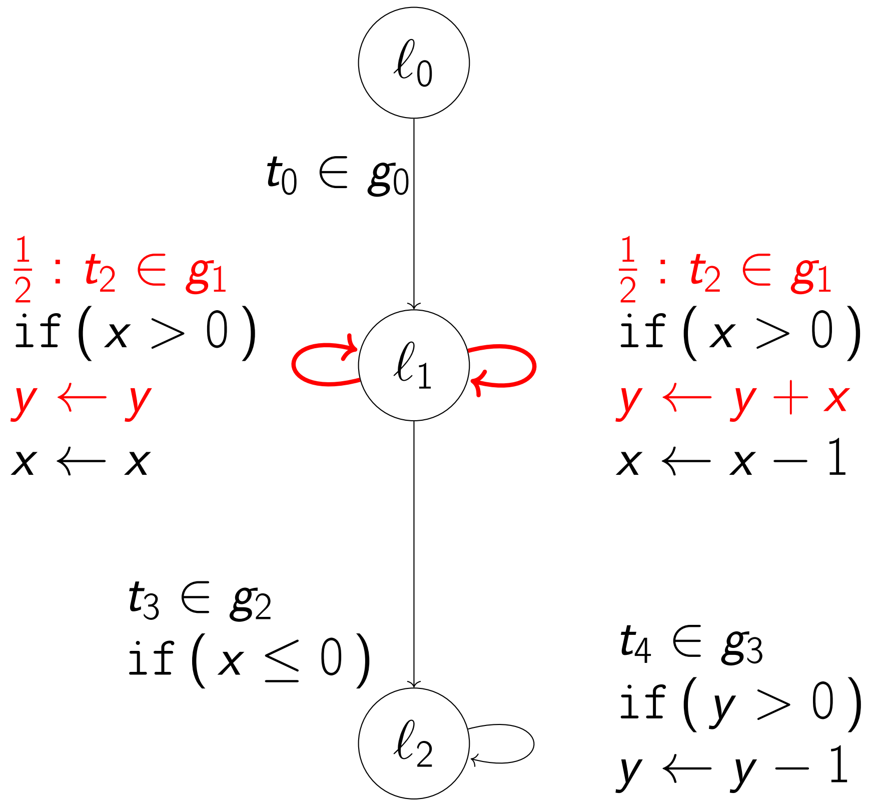
Computing Expected Size Bounds

Overall Idea

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\leq \mathbb{E}\big(\text{incoming-size}(y) + \mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)\big)$$

$$= \mathbb{E}\big(\text{incoming-size}(y)\big) + \mathbb{E}\big(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)\big)$$



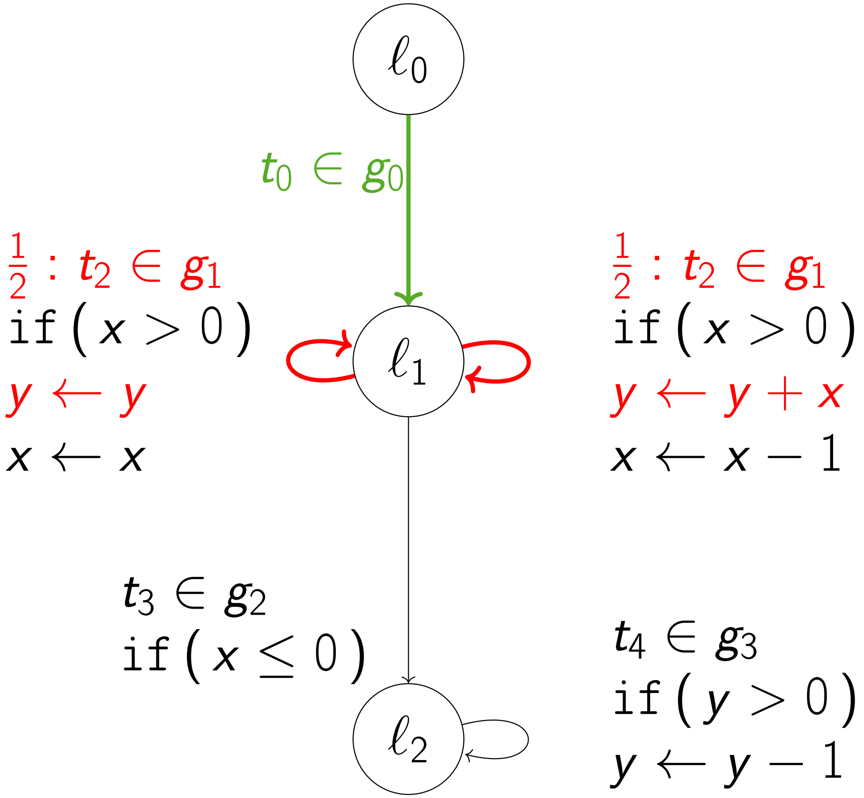
Computing Expected Size Bounds

Overall Idea

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\leq \mathbb{E}(\text{incoming-size}(y) + \mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y))$$

$$= \mathbb{E}(\text{incoming-size}(y)) + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y))$$

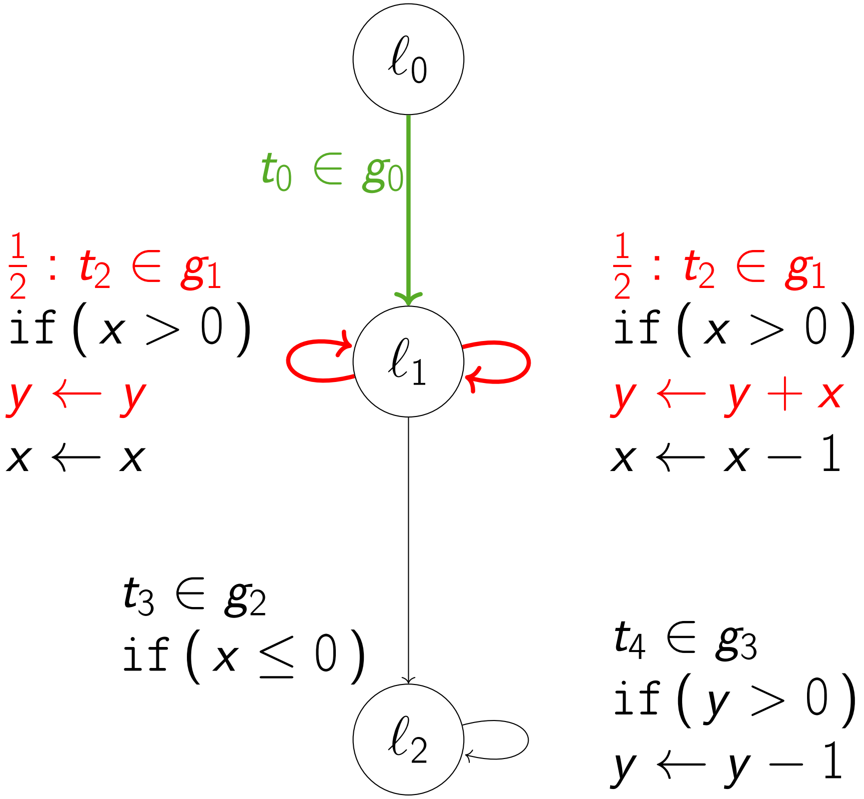


Computing Expected Size Bounds

Overall Idea

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\begin{aligned} &\leq \mathbb{E}\big(\text{incoming-size}(y) + \mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)\big) \\ &= y_0 + \mathbb{E}\big(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)\big) \end{aligned}$$

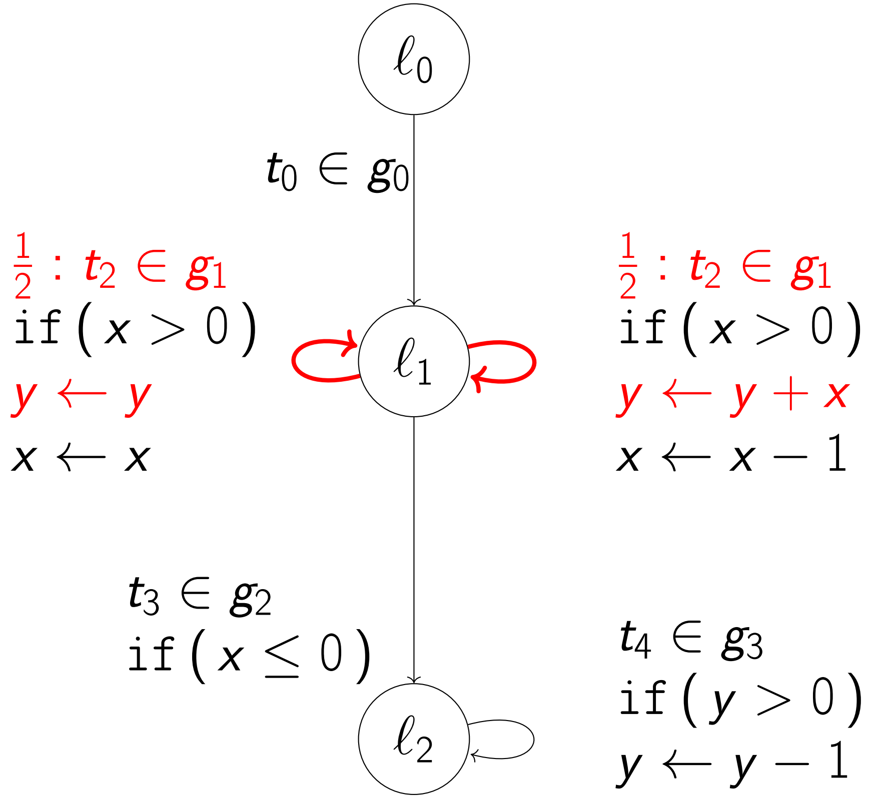


Computing Expected Size Bounds

Computation

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\leq \; y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y))$$



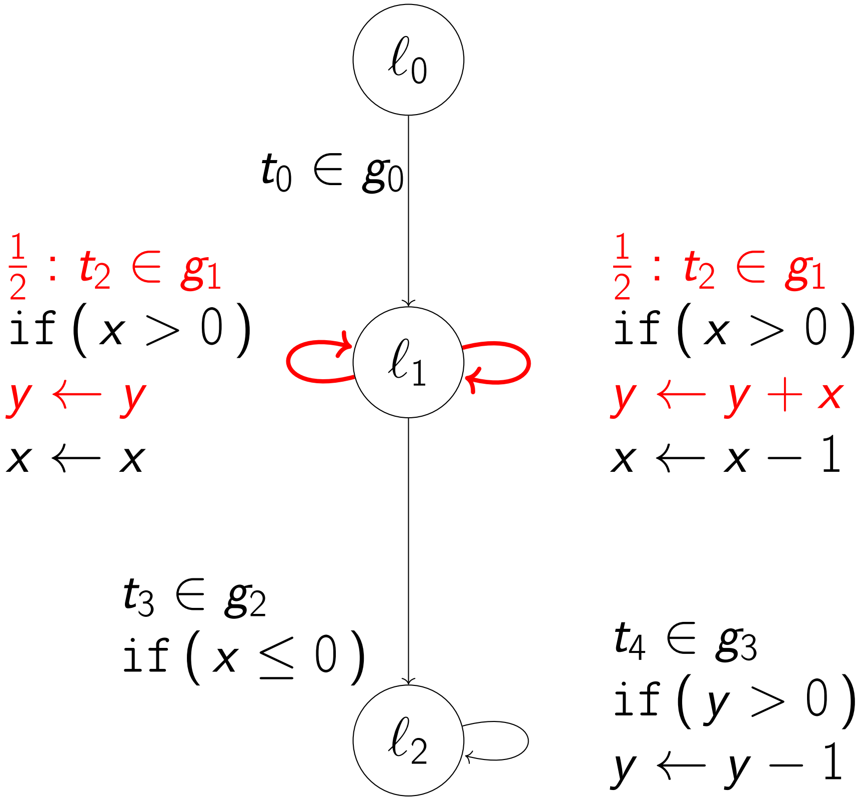
Computing Expected Size Bounds

Computation

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$\leq \; y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y))$

$\rightarrow$ worst-case-change is independent of runtime.

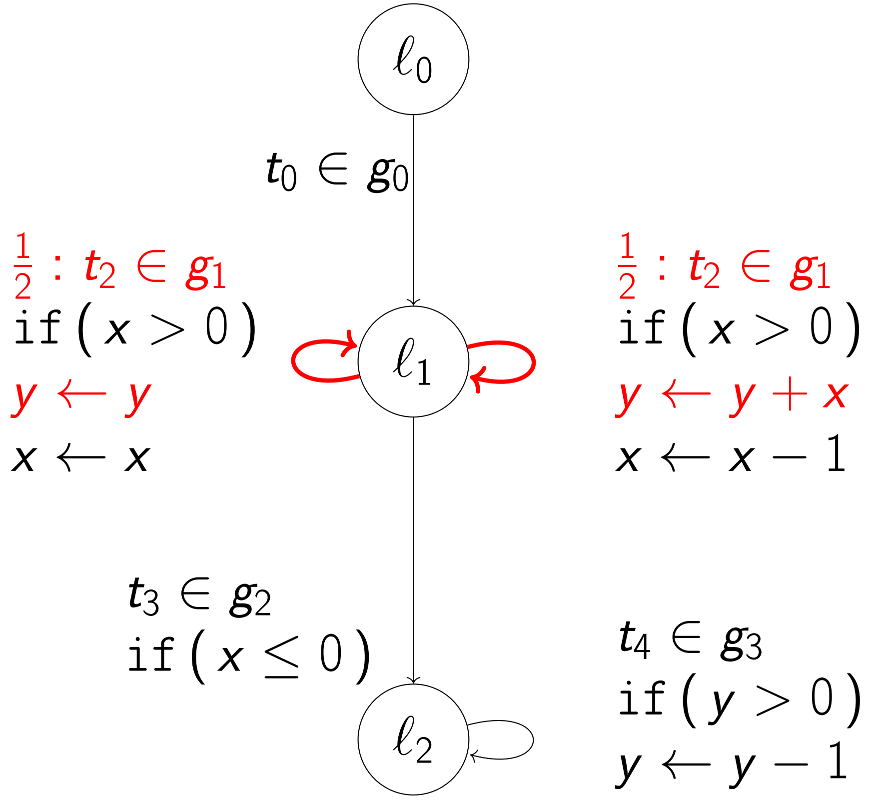


Computing Expected Size Bounds

Computation

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\leq \; y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y))$$

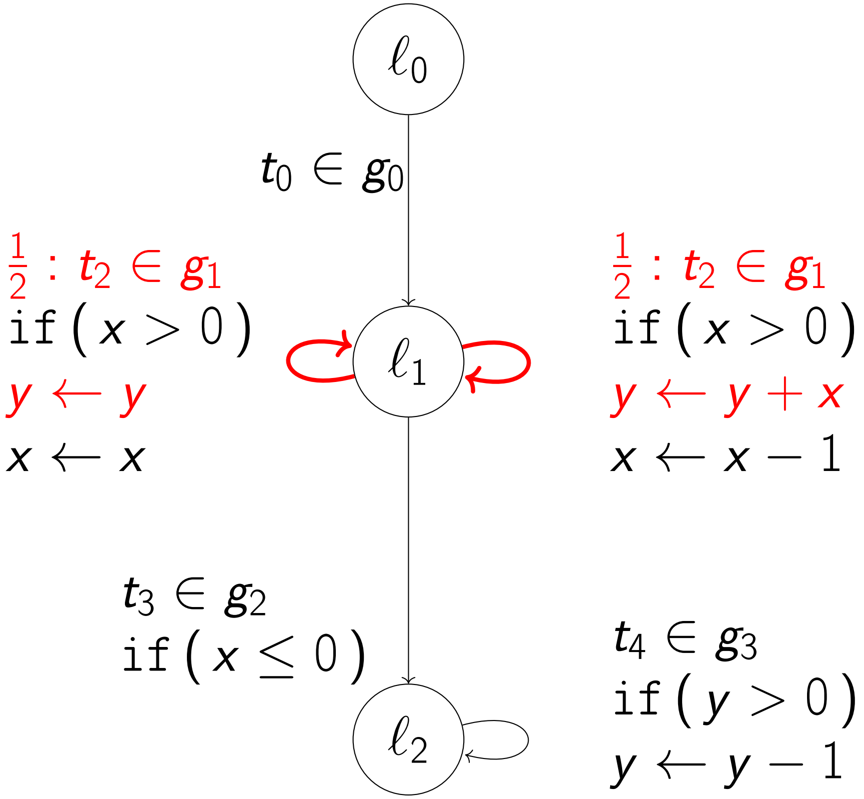


Computing Expected Size Bounds

Computation

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\begin{aligned}
&\le\ y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)) \\
&=\ y_0 + \mathbb{E}(\mathcal{R}(g_1)) \cdot \mathbb{E}(\text{worst-case-change}(g_1, y))
\end{aligned}$$

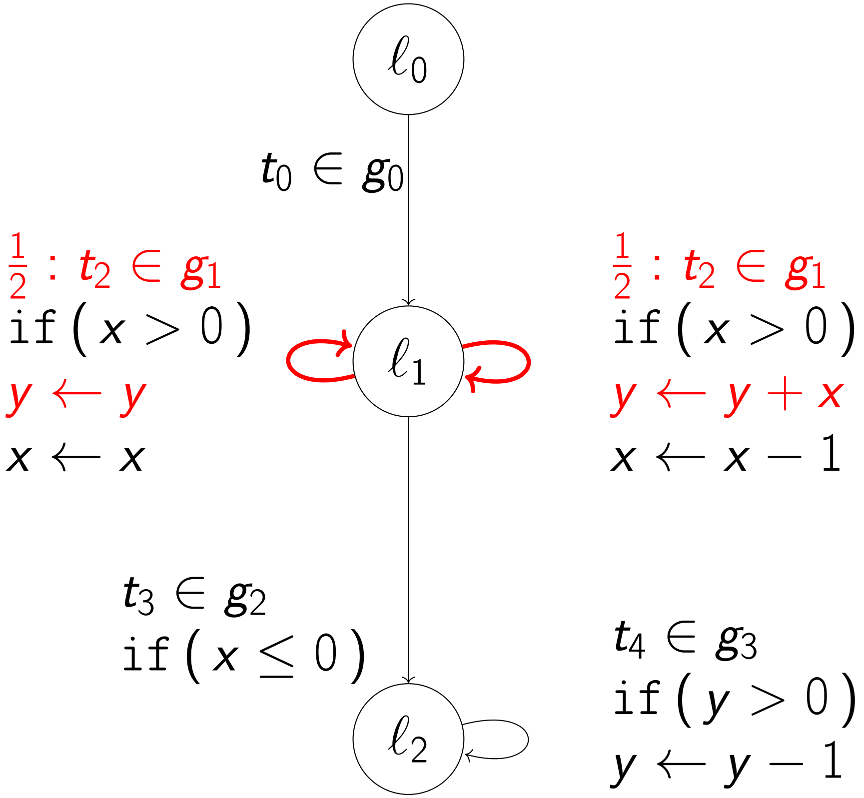


Computing Expected Size Bounds

Computation

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\begin{aligned} &\leq \; y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)) \\ &= \; y_0 + 2 \cdot x_0 \cdot \mathbb{E}(\text{worst-case-change}(g_1, y)) \end{aligned}$$

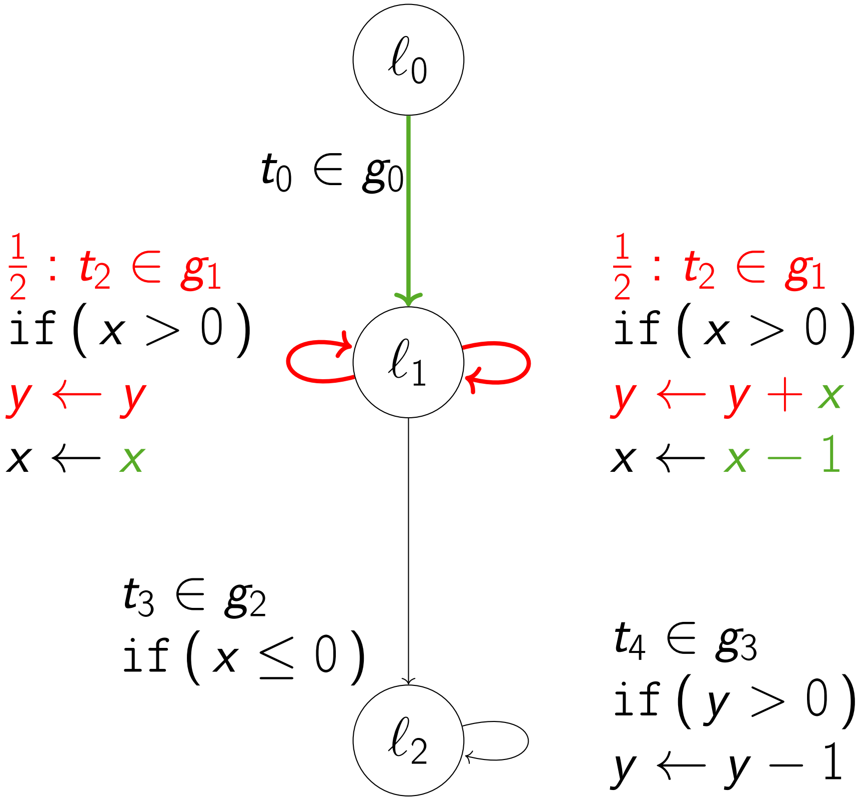


Computing Expected Size Bounds

Computation

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\begin{aligned} &\leq y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)) \\ &= y_0 + 2 \cdot x_0 \cdot \mathbb{E}(\text{worst-case-change}(g_1, y)) \end{aligned}$$

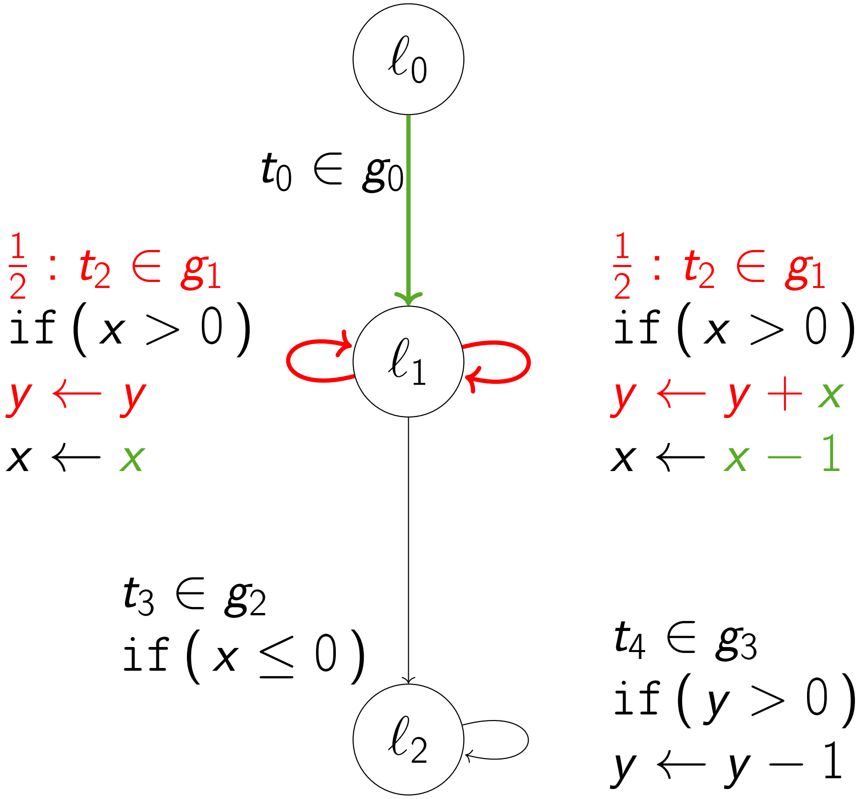


Computing Expected Size Bounds

Computation

Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\begin{aligned}&\leq\ y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y))\\ &=\ y_0 + 2 \cdot x_0 \cdot \tfrac{x_0}{2}\end{aligned}$$

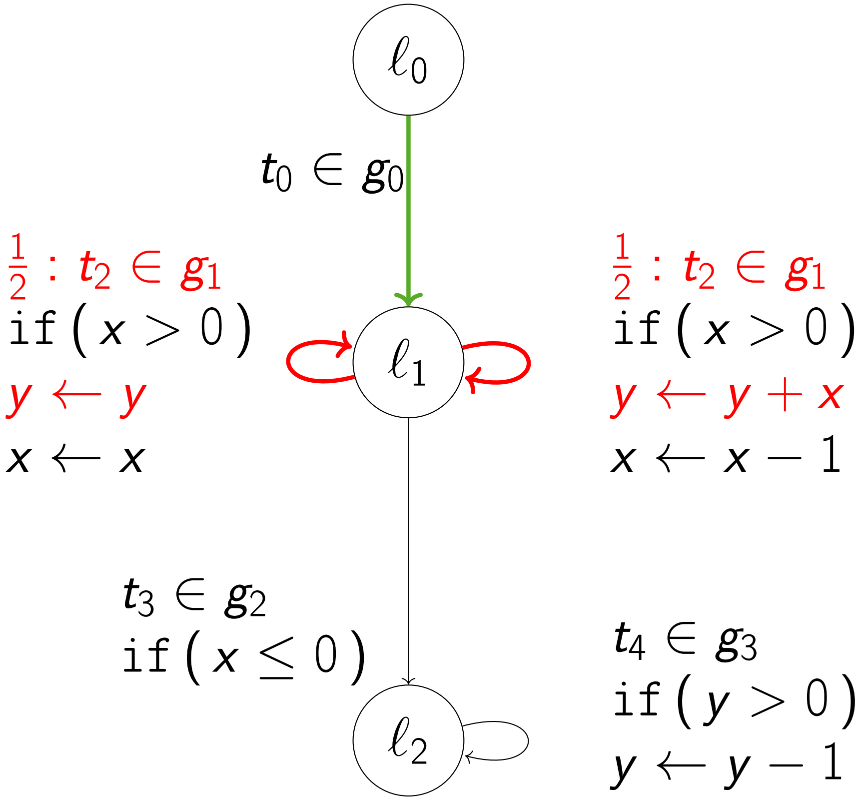


Computing Expected Size Bounds

Computation

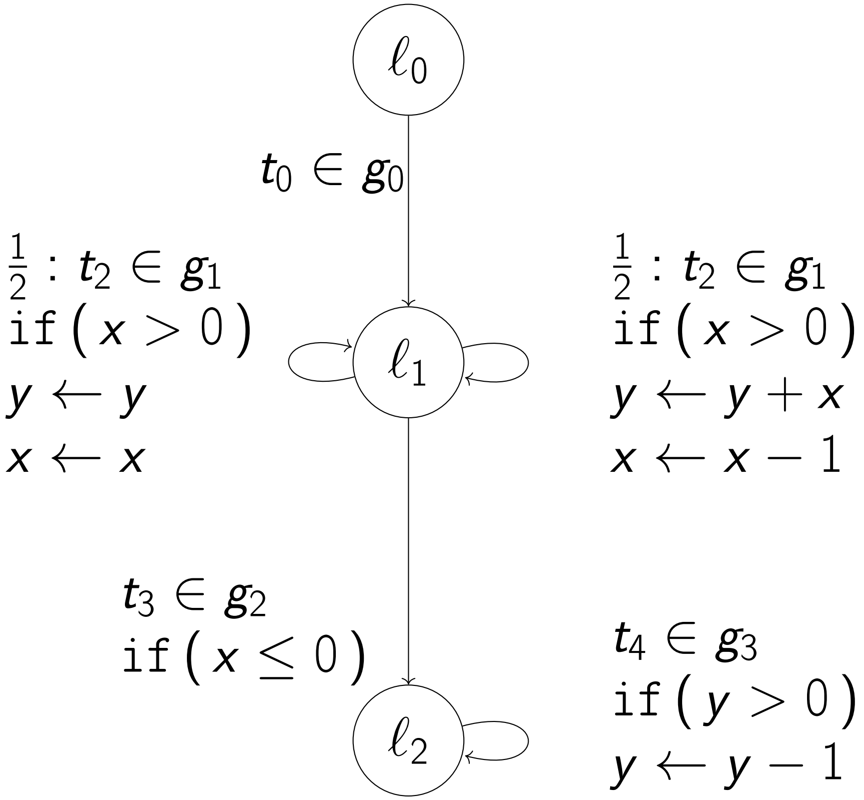
Expected size $\mathbb{E}(\mathcal{S}(g_1, y))$:

$$\begin{aligned} &\leq \; y_0 + \mathbb{E}(\mathcal{R}(g_1) \cdot \text{worst-case-change}(g_1, y)) \\ &= \; y_0 + x_0^2 \end{aligned}$$



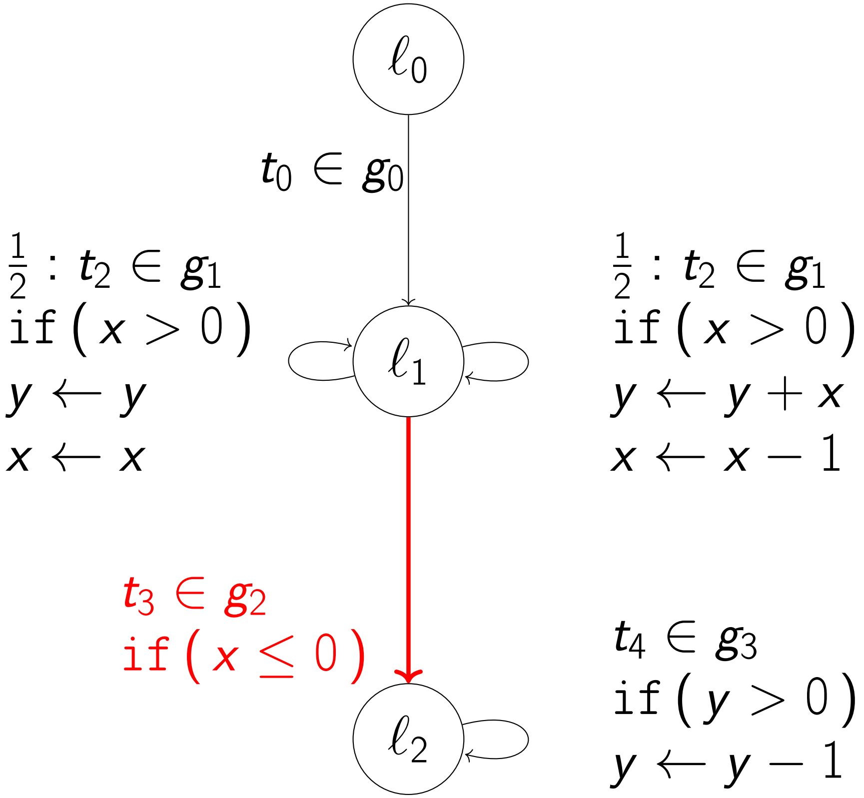
Computing Expected Size Bounds

Computation



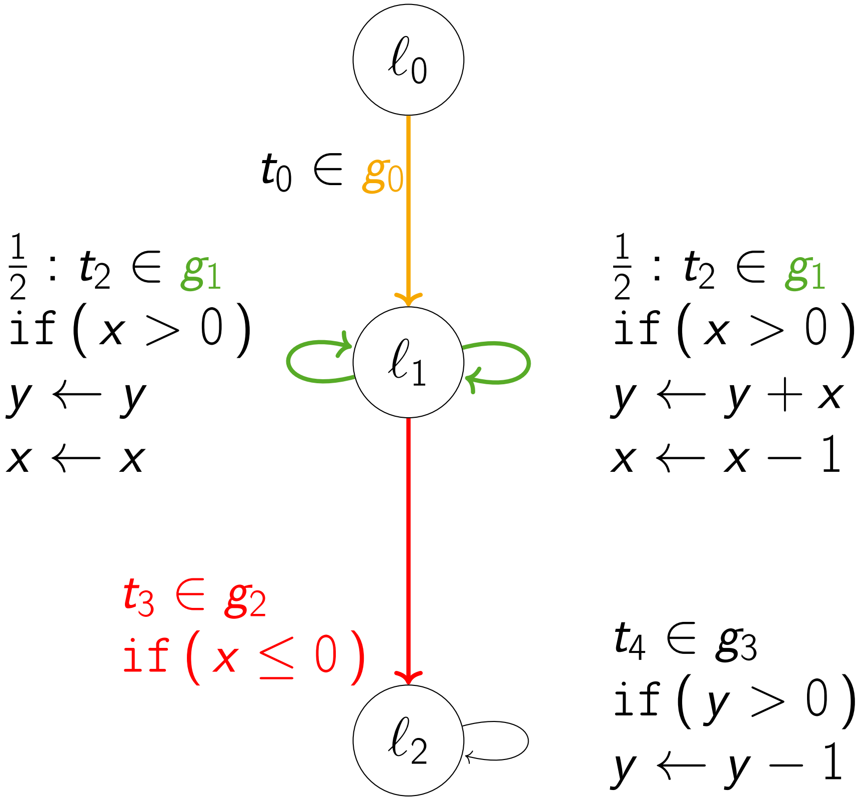
Computing Expected Size Bounds

Computation



Computation

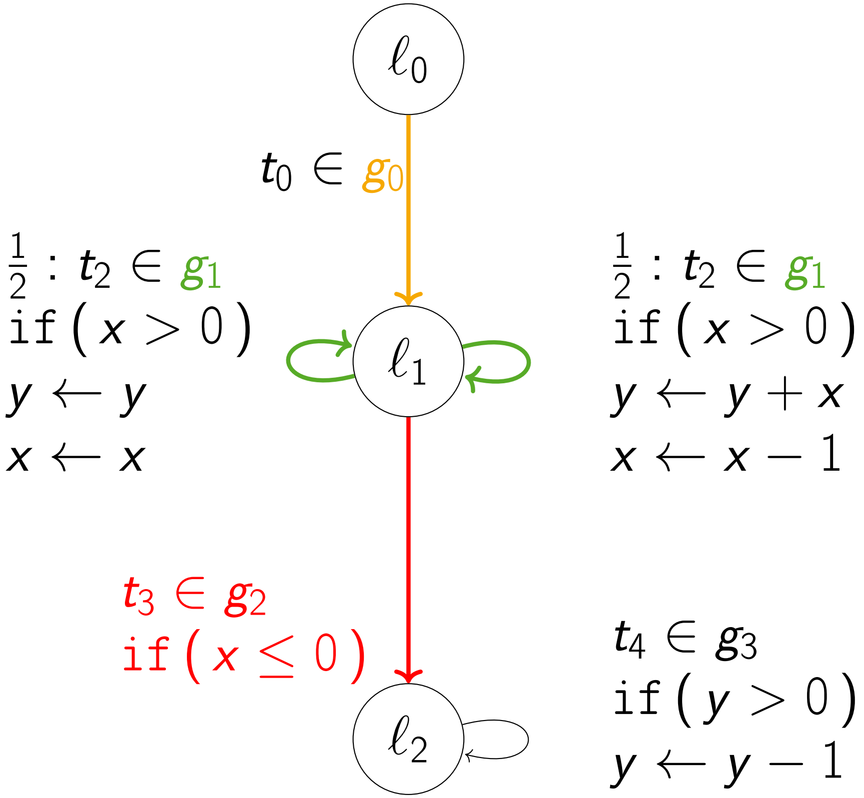
Expected size of v after g_2:



Computation

Expected size of v after g_2:

$\leq$ Maximal expected size of v after g_0, g_1.



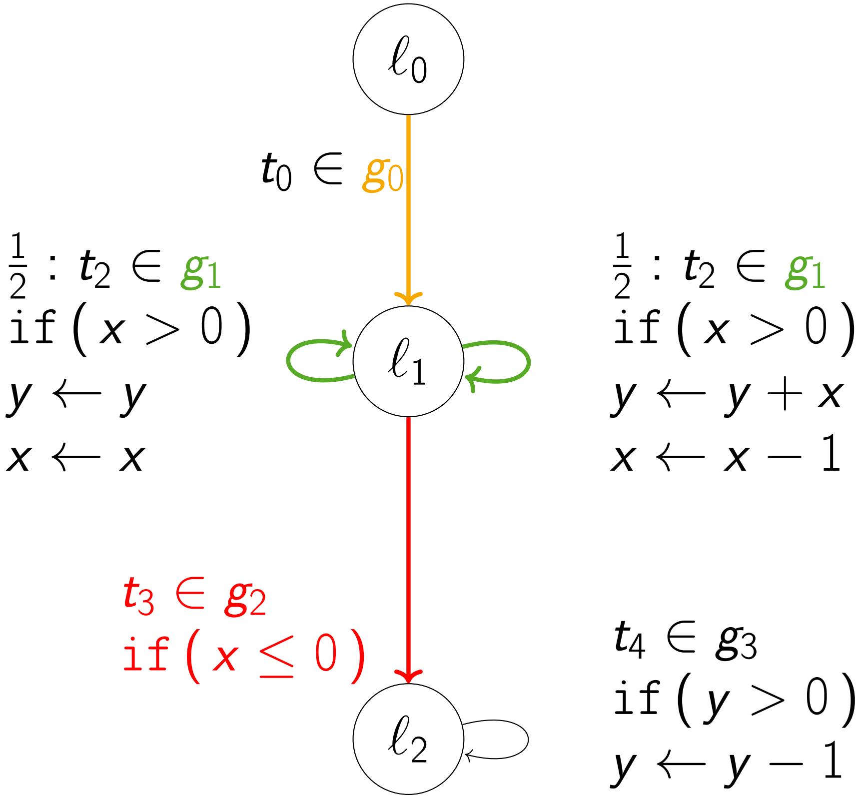
Computing Expected Size Bounds

Computation

Expected size of v after g_2:

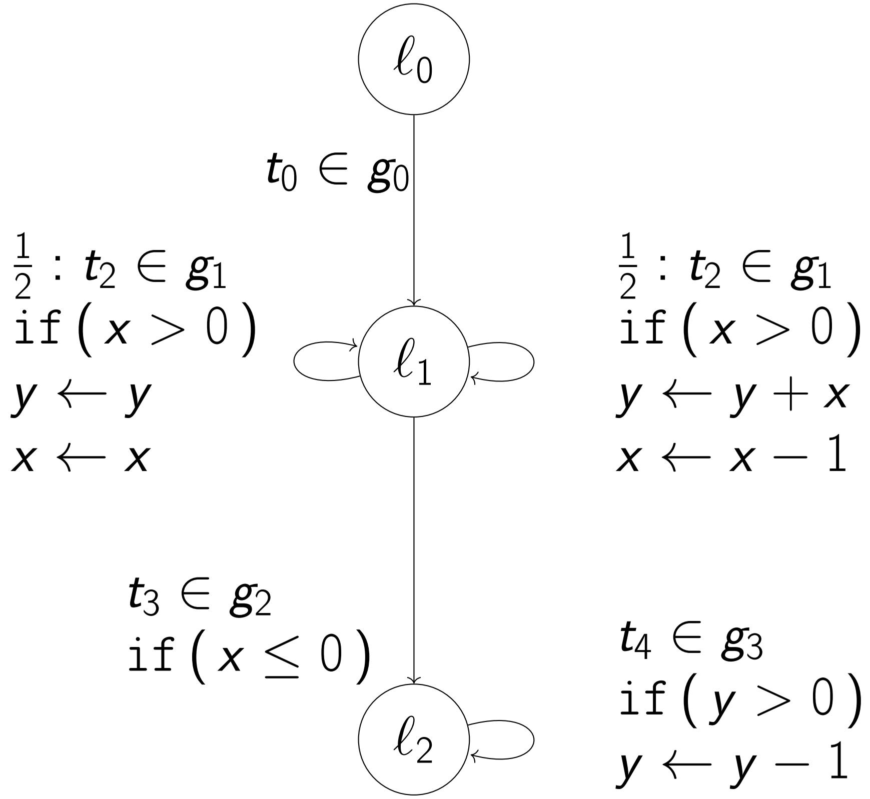
$\leq$ Maximal expected size of v after g_0, g_1.

$\to$ Expected size of y after g_2: $y_0 + x_0^2$.



Computing Expected Size Bounds

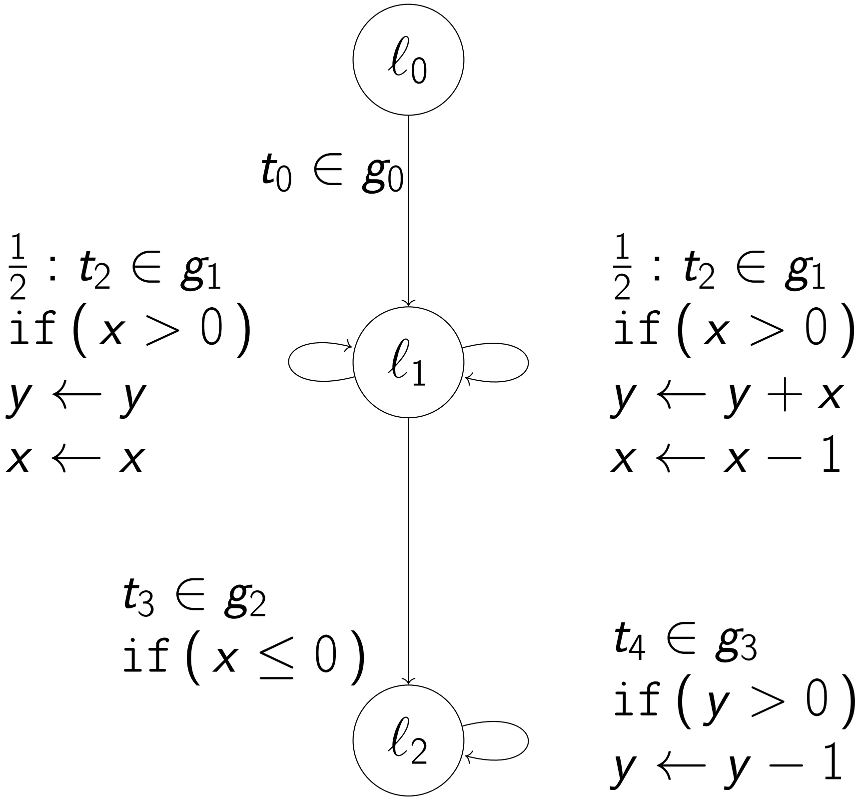
Summary



Computing Expected Size Bounds

Summary

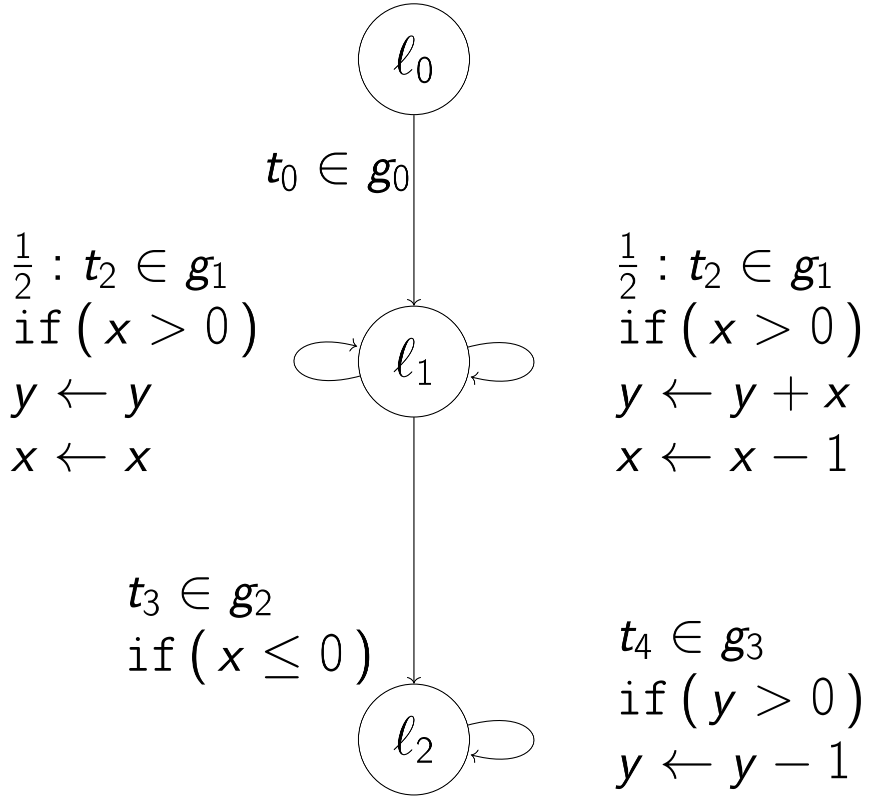
- Automatic inference of expected sizes:



Computing Expected Size Bounds

Summary

- Automatic inference of expected sizes:
  - $\rightarrow$ Incoming expected sizes.



Computing Expected Size Bounds

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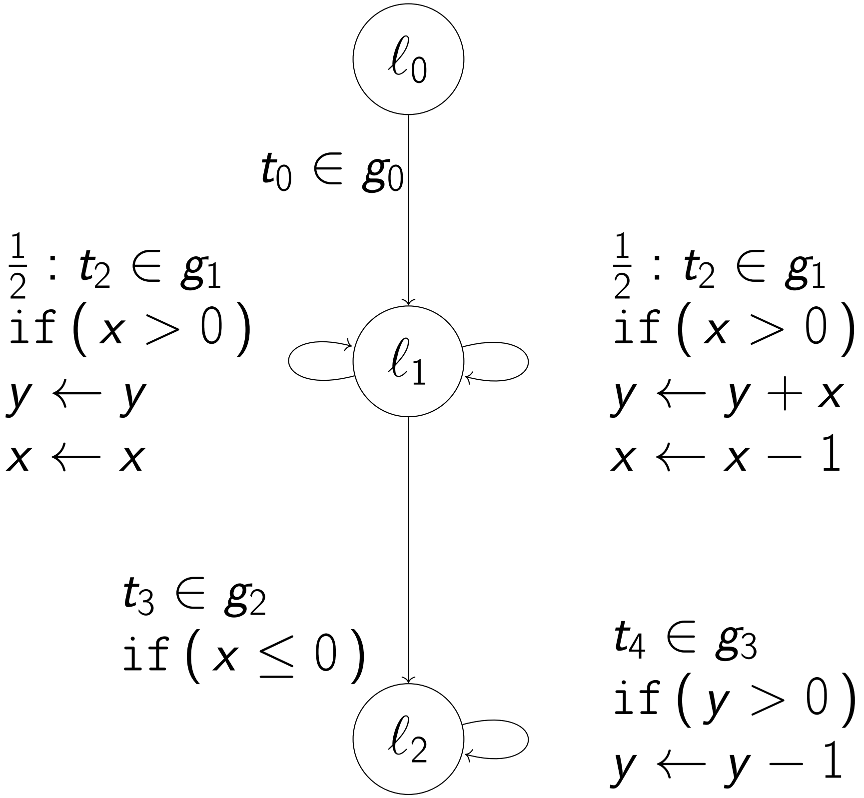

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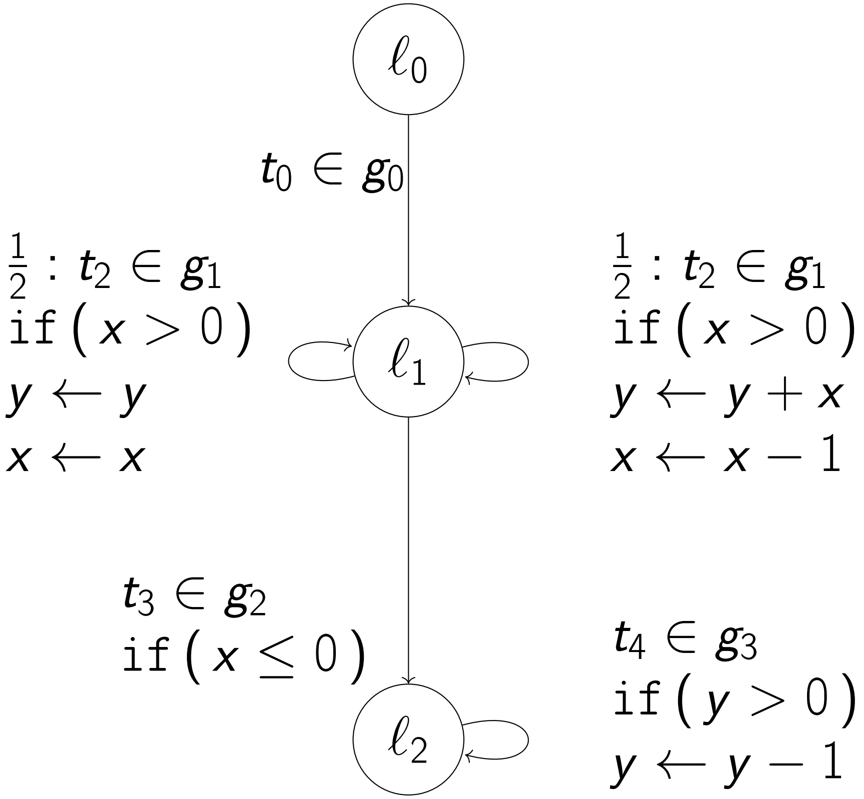

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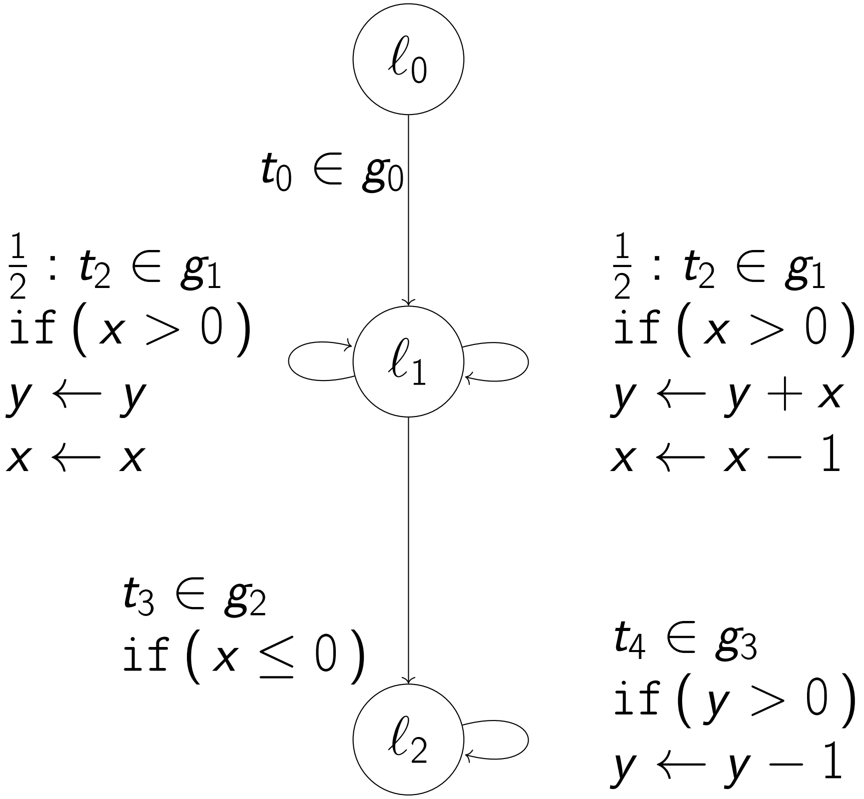

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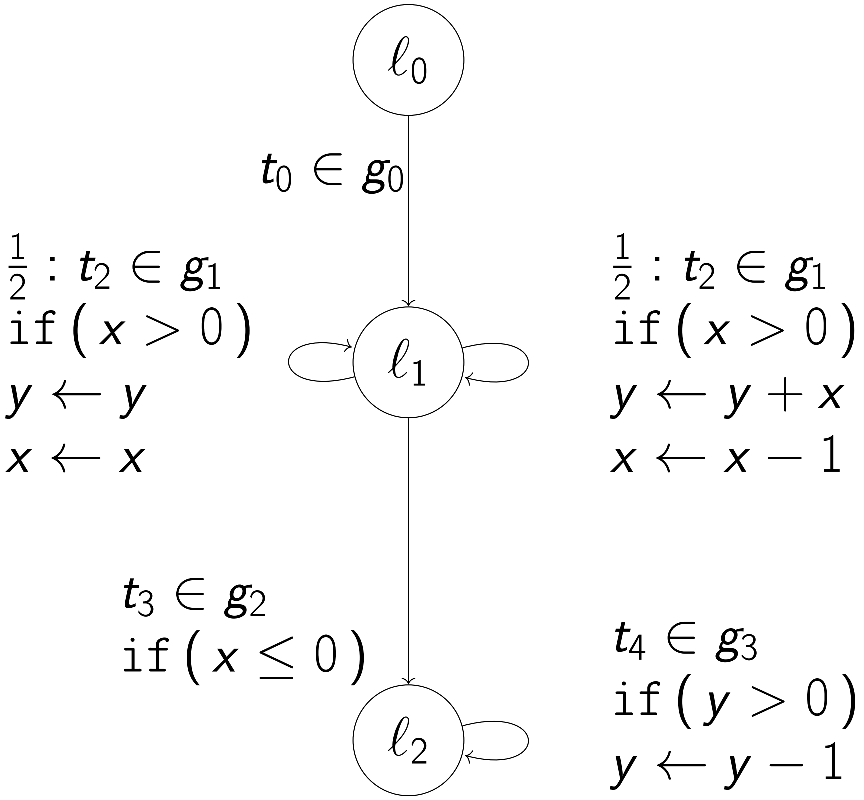

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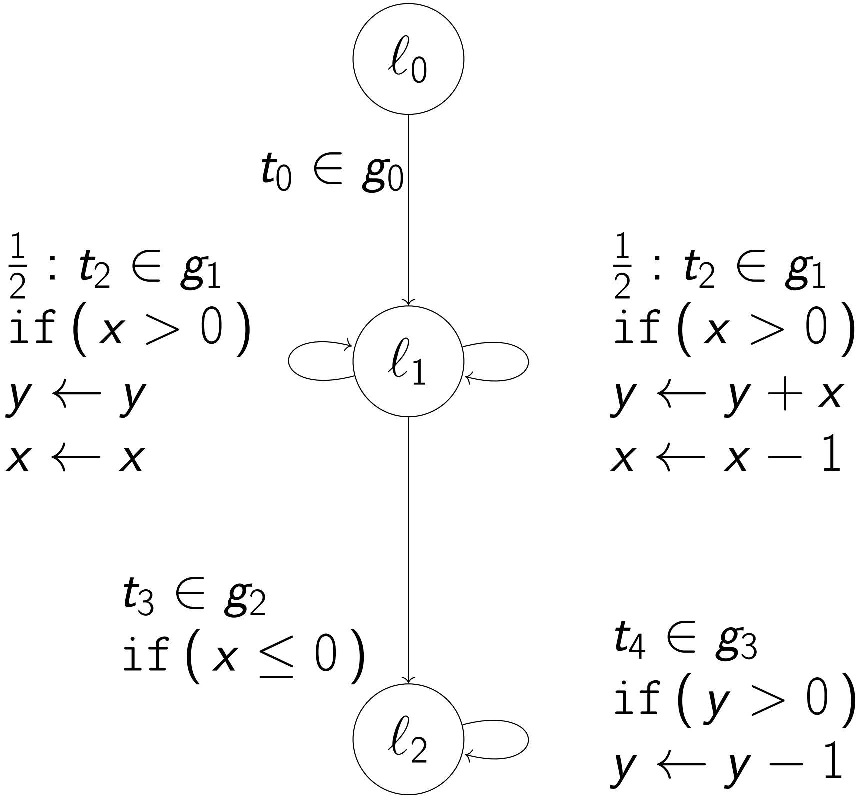

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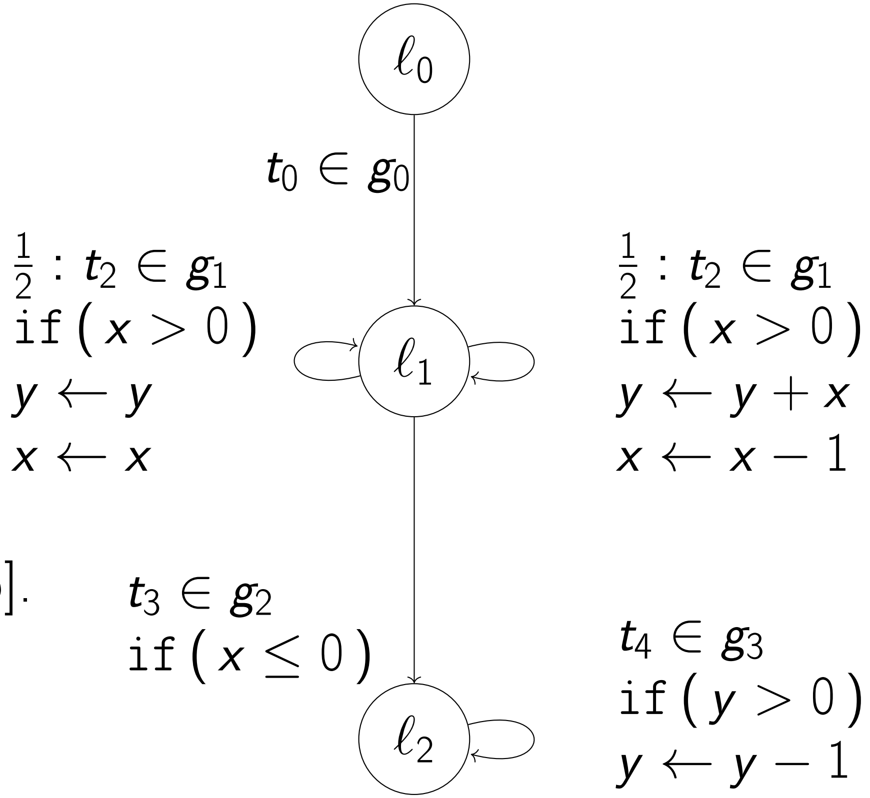
  - → Uses classical worst-case sizes for these variables [Giesl et al. '16].



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Thank you